CONSUMER CHOICE

1. THE CONSUMER CHOICE PROBLEM

1.1. **Unit of analysis and preferences.** The *fundamental unit of analysis* in economics is the *economic agent*. Typically this agent is an individual consumer or a firm. The agent might also be the manager of a public utility, the stockholders of a corporation, a government policymaker and so on.

The underlying assumption in economic analysis is that all economic agents possess a *preference ordering* which allows them to rank alternative states of the world.

The *behavioral* assumption in economics is that all agents make choices consistent with these underlying preferences.

1.2. **Definition of a competitive agent.** A buyer or seller (agent) is said to be competitive if the agent assumes or believes that the market price of a product is given and that the agent's actions do not influence the market price or opportunities for exchange.

1.3. **Commodities.** Commodities are the objects of choice available to an individual in the economic system. Assume that these are the various products and services available for purchase in the market. Assume that the number of products is finite and equal to L (ℓ =1, ..., L). A product vector is a list of the amounts of the various products:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{bmatrix}$$

The product bundle x can be viewed as a point in R^L.

1.4. **Consumption sets.** The consumption set is a subset of the product space R^L, denoted by $X^L \subset R^L$, whose elements are the consumption bundles that the individual can conceivably consume given the physical constraints imposed by the environment. We typically assume that the consumption set is $X = R^L_+ = \{x \in R^L\colon x_\ell \geq 0 \text{ for } \ell = 1, \ldots, L\}$.

1.5. **Prices.** We will assume that all L products are traded in the market at dollar prices that are publicly quoted. How they are determined will be discussed later. The prices are represented by a price vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix} \in R^L$$

For now assume that all prices are strictly positive, i.e. $p_\ell >> 0$. We will also assume that all consumers are price takers in the sense that they cannot influence the price at which they buy or sell a product.

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1.6. **Income or wealth.** Assume that each consumer has wealth equal to m_i or the representative consumer has wealth m.

1.7. **Affordable consumption bundles.** We say that a consumption bundle x is affordable for the representative consumer if

$$px = p_1 x_1 + p_2 x_2 + \cdots + p_{L x_L} \leq m \tag{1}$$

If x is also an element of R^L_+ , then the set of feasible consumption bundles is x ϵ R^L_+ : px $\leq$ m. This is called a Walrasian budget set and is denoted $B_{p,m}$.

1.8. **Preferences.** We assume a preference relation over products $\succeq$ with the following properties

1: complete in that for all x_1, $x_2 \in X$, we have $x_1 \succeq x_2$ or $x_2 \succeq x_1$ (or both)
2: transitive in that $\forall$ x_1, x_2, x_3 ϵ X, if $x_1 \succeq x_2$ and $x_2 \succeq x_3$ then $x_1 \succeq x_3$.
3: locally nonsatiated in that for every $x_1 \in X$ and every $\varepsilon > 0$, there is $x_2 \in X$ such that $||x_2 - x_1 \leq \varepsilon||$ and $x_2 \succeq x_1$.
4: continuous in that for any sequence of pairs

$$\{ (x_1^n, x_2^n) \}_{n=1}^{\infty} \; with \; x_1^n \succeq x_2^n \forall n,$$
$$x_1 = \lim_{n \to \infty} x_1^n, \; and \; x_2 = \lim_{n \to \infty} x_2^n,$$
we have $x_1 \succeq x_2$.

1.9. **Existence of a utility function.** Based on the preferences defined in 1.8, there exists a continuous utility function v(x) that represents $\succeq$ in the sense that $x_1 \succeq x_2$ iff $v(x_1) \geq v(x_2)$.

1.10. **Convexity.** We often assume that preferences are convex in the sense that if $x_1 \succeq x_2$, then for $0 \leq \lambda \leq 1$, $\lambda x_1 + (1-\lambda) x_2 \succeq x_1$. This implies that indifference curves are convex to the origin. If the utility function is quasi-concave, then the indifference curves will be convex and vice versa.

2. The utility maximization problem

2.1. **Formal Problem.** The utility maximization problem for the consumer is then as follows

$$\begin{aligned} \max_{x \geq 0} \; u \; &= v(x) \\ s.t. \; px \; &\leq \; m \end{aligned} \tag{2}$$

where we assume that p >> 0, m > 0 and X = R^L_+ .

This is called the *primal* preference problem. If we have smooth convex indifference curves and an interior solution, then the problem can be solved using standard Lagrangian techniques. Alternatively, Kuhn-Tucker methods can be used. The Lagrangian function is given by

$$\mathcal{L} = v(x) - \lambda (\Sigma_{i=1}^n p_i x_i - m) \tag{3}$$

The first order conditions are

$$\begin{aligned} \frac{\partial v}{\partial x_i} - \lambda p_i = 0, \quad i = 1, 2, \ldots, n \\ - \Sigma_{i=1}^n p_i x_i + m = 0 \end{aligned} \tag{4}$$

The value of λ is the amount by which $\mathcal{L}$ would increase given a unit relaxation in the constraint (an increase in income). It can be interpreted as the marginal utility of expenditure. This units of this are of course arbitrary. The solution to 2 is given by x(p,m) = g(p,m). These functions are called Marshallian demand equations. Note that they depend on the prices of all good and income. Based on the structure of preferences and the nature of the optimization problem, they will have certain properties which we will discuss shortly.

2.2. **Cobb-Douglas Example.** Consider a utility function given by

$$u = v(x) = x_1^{\alpha_1} x_2^{\alpha_2} \tag{5}$$

We usually assume that $\alpha_i > 0$. To maximize utility subject to a budget constraint we obtain we set up a Lagrangian function.

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda \left[p_1 x_1 + p_2 x_2 - m \right] \tag{6}$$

Differentiating equation 6 we obtain

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\alpha_1 x_1^{\alpha_1} x_2^{\alpha_2}}{x_1} - \lambda p_1 = 0 \tag{7a}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2}}{x_2} - \lambda p_2 = 0 \tag{7b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left[p_1 x_1 + p_2 x_2 \right] + m = 0 \tag{7c}$$

Take the ratio of the 7a and 7b to obtain

$$\begin{aligned} \frac{\frac{\alpha_1 x_1^{\alpha_1} x_2^{\alpha_2}}{x_1}}{\frac{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2}}{x_2}} &= \frac{p_1}{p_2} \\ \Rightarrow \frac{\alpha_1 x_2}{\alpha_2 x_1} &= \frac{p_1}{p_2} \end{aligned} \tag{8}$$

We can now solve the equation for the quantity of good 2 as a function of the quantity of good 1 and the prices of both goods. Doing so we obtain

$$x_2 = \frac{\alpha_2 x_1 p_1}{\alpha_1 p_2} \tag{9}$$

Now substitute 9 in 7c to obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} &= -\left[p_1 x_1 + p_2 x_2 \right] + m = 0 \\ \Rightarrow p_1 x_1 + p_2 \frac{\alpha_2 x_1 p_1}{\alpha_1 p_2} &= m \\ \Rightarrow \frac{p_1 \alpha_1 x_1}{\alpha_1} + \frac{\alpha_2 x_1 p_1}{\alpha_1} &= m \\ \Rightarrow \frac{p_1 x_1}{\alpha_1} (\alpha_1 + \alpha_2) &= m \\ \Rightarrow x_1 &= \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \frac{m}{p_1} \end{aligned} \tag{10}$$

Similarly for x_2 so that we have

$$x_2 \;=\; \frac{\alpha_2}{(\alpha_1 \;+\; \alpha_2\,)}\,\frac{m}{p_2} \tag{11}$$

Note that demand for the kth good only depends on the kth price and is homogeneous of degree zero in prices and income. Also note that it is linear in income. This implies that the expenditure elasticity is equal to 1. This can be seen as follows.

$$\begin{aligned} x_1 \;&=\frac{\alpha_1}{(\alpha_1 \;+\; \alpha_2\,)}\,\frac{m}{p_1} \\ \Rightarrow\; \frac{\partial\, x_1}{\partial\, m}\,\frac{m}{x_1} \;&=\; \left[\frac{\alpha_1}{(\alpha_1 \;+\; \alpha_2\,)}\,\frac{1}{p_1}\right]\;\left[\frac{m}{\frac{\alpha_1}{(\alpha_1\,+\,\alpha_2\,)}\,\frac{m}{p_1}}\right] \;=\; 1 \end{aligned} \tag{12}$$

We can find the value of the optimal u by substitution

$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left(\frac{m}{p_1}\left[\frac{\alpha_1}{\alpha_1+\alpha_2}\right]\right)^{\alpha_1}\left(\frac{m}{p_2}\left[\frac{\alpha_2}{\alpha_1+\alpha_2}\right]\right)^{\alpha_2} \\ &= m^{\alpha_1+\alpha_2} p_1^{-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(\alpha_2+\alpha_2)^{-\alpha_1-\alpha_2} \end{aligned} \tag{13}$$

This can also be written

$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left[\frac{m}{p_1}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)\right]^{\alpha_1}\left[\frac{m}{p_2}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\right]^{\alpha_2} \\ &= \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{\alpha_1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^{\alpha_2}\left(\frac{m}{p_1}\right)^{\alpha_1}\left(\frac{m}{p_2}\right)^{\alpha_2} \end{aligned} \tag{14}$$

For future reference note that the derivative of the optimal u with respect to m is given by

$$\begin{aligned} u &= m^{\alpha_1+\alpha_2} p_1^{-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(\alpha_2+\alpha_2)^{-\alpha_1-\alpha_2} \\ \frac{\partial u}{\partial m} &= (\alpha_1+\alpha_2) m^{\alpha_1+\alpha_2-1} p_1^{-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1}\alpha_1^{\alpha_2}(\alpha_2+\alpha_2)^{-\alpha_1-\alpha_2} \\ &= m^{\alpha_1+\alpha_2-1} p_1^{-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1}\alpha_1^{\alpha_2}(\alpha_2+\alpha_2)^{1-\alpha_1-\alpha_2} \end{aligned} \tag{15}$$

We obtain λ by substituting in either the first or second equation as follows

$$\begin{aligned} \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} - \lambda p_1 &= 0 \\ \Rightarrow \lambda &= \frac{\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2}}{p_1} \\ \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} - \lambda p_2 &= 0 \\ \Rightarrow \lambda &= \frac{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1}}{p_2} \end{aligned} \tag{16}$$

If we now substitute for x_1 and x_2, we obtain

$$
\begin{aligned}
\lambda &= \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{p_1} \\
x_1 &= \frac{m}{p_1}\left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right] \\
x_2 &= \frac{m}{p_2}\left[\frac{\alpha_2}{\alpha_1 + \alpha_2}\right] \\
\Rightarrow \lambda &= \frac{\alpha_1 \left(\frac{m}{p_1}\left[\frac{\alpha_1}{\alpha_1 + \alpha_2}\right]\right)^{\alpha_1 - 1} \left(\frac{m}{p_2}\left[\frac{\alpha_2}{\alpha_1 + \alpha_2}\right]\right)^{\alpha_2}}{p_1} \\
&= \frac{\alpha_1 m^{\alpha_1 + \alpha_2 - 1} p_1^{1-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1 - 1} \alpha_2^{\alpha_2} \left(\alpha_1 + \alpha_2\right)^{1 - \alpha_1 - \alpha_2}}{p_1} \\
&= m^{\alpha_1 + \alpha_2 - 1} p_1^{-\alpha_1} p_2^{-\alpha_2} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \left(\alpha_1 + \alpha_2\right)^{1 - \alpha_1 - \alpha_2}
\end{aligned}
\tag{17}
$$

Thus λ is equal to the derivative of the optimal u with respect to m.

To check for a maximum or minimum we set up the bordered Hessian. The bordered Hessian in this case is

$$
H_B = \begin{bmatrix}
\frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_1 \partial x_2} & \frac{\partial g(x^*)}{\partial x_1} \\
\frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x^*,\, \lambda^*)}{\partial x_2 \partial x_2} & \frac{\partial g(x^*)}{\partial x_2} \\
\frac{\partial g(x^*)}{\partial x_1} & \frac{\partial g(x^*)}{\partial x_2} & 0
\end{bmatrix}
\tag{18}
$$

We compute the various elements of the bordered Hessian as follows

$$\mathcal{L} = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda[p_1 x_1 + p_2 x_2 - m]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda p_1$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} - \lambda p_2$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1^2} = (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1}$$

$$\frac{\partial^2 \mathcal{L}}{\partial x_2^2} = (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2}$$

$$\frac{\partial g}{\partial x_1} = p_1$$

$$\frac{\partial g}{\partial x_2} = p_2$$

The derivatives of the constraints are constants. The bordered Hessian is given by

$$H_B = \begin{bmatrix} (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} & \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & p_1 \\ \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} & p_2 \\ p_1 & p_2 & 0 \end{bmatrix} \tag{19}$$

To find the determinant of the bordered Hessian expand by the third row as follows

$$\begin{aligned} |H_B| &= (-1)^4 p_1 \begin{vmatrix} \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & p_1 \\ (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} & p_2 \end{vmatrix} + (-1)^5 p_2 \begin{vmatrix} (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} & p_1 \\ \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & p_2 \end{vmatrix} + 0 \\ &= p_1 \begin{vmatrix} \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & p_1 \\ (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} & p_2 \end{vmatrix} - p_2 \begin{vmatrix} (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} & p_1 \\ \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} & p_2 \end{vmatrix} \\ &= p_1 p_2 \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} - p_1^2 (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} \\ &\qquad - p_2^2 (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} + p_1 p_2 \alpha_1 \alpha_2 x_1^{\alpha_1 - 1} x_2^{\alpha 2 - 1} \\ &= 2 p_1 p_2 \alpha_1 \alpha 2 x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} - p_1^2 (\alpha_2)(\alpha_2 - 1) x_1^{\alpha_1} x_2^{\alpha_2 - 2} - p_2^2 (\alpha_1)(\alpha_1 - 1) x_1^{\alpha_1 - 2} x_2^{\alpha_2} \end{aligned} \tag{20}$$

For a maximum we want equation 20 to be positive. Rewriting it we obtain

$$2p_1p_2\alpha_1\alpha_2x_1^{\alpha_1-1}x_2^{\alpha_2-1} - p_1^2(\alpha_2)(\alpha_2-1)x_1^{\alpha_1}x_2^{\alpha_2-2} - p_2^2(\alpha_1)(\alpha_1-1)x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \tag{21}$$

We can also write it in the following convenient way

$$\begin{gathered} 2p_1p_2\alpha_1\alpha_2x_1^{\alpha_1-1}x_2^{\alpha_2-1} \\ +\alpha_2p_1^2x_1^{\alpha_1}x_2^{\alpha_2-2} - \alpha_2^2p_1^2x_1^{\alpha_1}x_2^{\alpha_2-2} \\ +\alpha_1p_2^2x_1^{\alpha_1-2}x_2^{\alpha_2} - \alpha_1^2p_2^2x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \end{gathered} \tag{22}$$

To eliminate the prices we can substitute from the first-order conditions.

$$p_1 = \frac{\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}$$

$$p_2 = \frac{\alpha_2x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}$$

This then gives

$$\begin{gathered} 2\left(\frac{\alpha_1x_1^{\alpha 1-1}x_2^{\alpha_2}}{\lambda}\right)\left(\frac{\alpha_2x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)\alpha_1\alpha_2x_1^{\alpha_1-1}x_2^{\alpha_2-1} \\ +\alpha_2\left(\frac{\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}\right)^2x_1^{\alpha_1}x_2^{\alpha_2-2} - \alpha_2^2\left(\frac{\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}}{\lambda}\right)^2x_1^{\alpha_1}x_2^{\alpha_2-2} \\ +\alpha_1\left(\frac{\alpha_2x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)^2x_1^{\alpha_1-2}x_2^{\alpha_2} - \alpha_1^2\left(\frac{\alpha_2x_1^{\alpha_1}x_2^{\alpha_2-1}}{\lambda}\right)^2x_1^{\alpha_1-2}x_2^{\alpha_2} > 0 \end{gathered} \tag{23}$$

Multiply both sides by λ^2 and combine terms to obtain

$$\begin{gathered} 2\alpha_1^2\alpha_2^2x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} \\ +\alpha_1^2\alpha_2x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} - \alpha_2^2\alpha_1^2x_1^{3\alpha 1-2}x_2^{3\alpha_2-2} \\ +\alpha_1\alpha_2^2x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} - \alpha_1^2\alpha_2^2x_1^{3\alpha_1-2}x_2^{3\alpha_2-2} > 0 \end{gathered} \tag{24}$$

Now factor out $x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}$ to obtain

$$\begin{aligned} x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}\left(2\alpha_1^2\alpha 2^2 + \alpha_1^2\alpha_2 - \alpha_2^2\alpha_1^2 + \alpha_1\alpha_2^2 - \alpha_1^2\alpha_2^2\right) &> 0 \\ \Rightarrow x_1^{3\alpha_1-2}x_2^{3\alpha_2-2}\left(\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2\right) &> 0 \end{aligned} \tag{25}$$

With positive values for x_1 and x_2 the whole expression will be positive if the last term in parentheses is positive. Then rewrite this expression as

$$\left(\alpha_1^2\alpha_2 + \alpha_1\alpha 2^2\right) > 0 \tag{26}$$

Now divide both sides by $\alpha_1^2\alpha_2^2$ (which is positive) to obtain

$$\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_1}\right) > 0 \tag{27}$$

3. THE EXPENDITURE (COST) MINIMIZATION PROBLEM

3.1. **Basic duality formulation.** The fundamental (primal) utility maximization problem is given by

$$\begin{aligned} \max_{x \geq 0} \, u \; &= v(x) \\ s.t. \, px \; &\leq \; m \end{aligned} \tag{28}$$

Dual to the utility maximization problem is the cost minimization problem

$$\begin{aligned} \min_{x \geq 0} \, m \; &= px \\ s.t. \, v(x) \; &= \; u \end{aligned} \tag{29}$$

3.2. **Marshallian and Hicksian demand functions.** The solution to equation 29 gives the Hicksian demand functions x = h(u, p). The Hicksian demand equations are sometimes called ”compensated” demand equations because they hold u constant. The solutions to the primal and dual problems coincide in the sense that

$$x \; = \; g\,(p, \; m) \; = \; h\,(u, \; p) \tag{30}$$

3.3. **Indirect objective functions.** We can substitute the optimal levels of the decision variables as functions of the parameters back into the objective functions to obtain the indirect objective functions. For the primal problem this gives

$$u \; = \; v\,(x_1, \; x_2, \; \cdots, \; x_n\,) \; = \; v\,[g_1\,(m, \; p), \; g_2\,(m, \; p), \; \ldots, \; g_n\,(m, \; p)\;] \; = \; \psi(m, \; p) \tag{31}$$

This is called the indirect utility function and specifies utility as a function of prices and income. We can also write it as follows

$$\psi\,(m, \; p) \; = \; \max_{x}\,[v(x) : \, px \; = \; m] \tag{32}$$

The indirect utility function for the Cobb-Douglas utility function is given by

$$\begin{aligned} u &= x_1^{\alpha_1} x_2^{\alpha_2} \\ &= \left[\frac{m}{p_1}\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)\right]^{\alpha_1} \left[\frac{m}{p_2}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\right]^{\alpha_2} \\ &= \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)^{\alpha_2} \left(\frac{m}{p_1}\right)^{\alpha_1} \left(\frac{m}{p_2}\right)^{\alpha_2} \end{aligned} \tag{33}$$

For the Cobb-Douglas utility function with multiple inputs, the indirect utility function is given by

$$
\begin{aligned}
\psi \;=\; v\,(x\,(p,\; m)\,) \;&=\; \prod_{i=1}^{n} x_i^{\alpha_i} \\
&= \prod_{i=1}^{n} \left[\frac{\alpha_i}{\Sigma_{j=1}^{n}\; \alpha_j} \; \frac{m}{p_i} \right]^{\alpha_i} \\
&= \prod_{i=1}^{n} \left[\frac{m}{\Sigma_{j=1}^{n}\; \alpha_j} \right]^{\alpha_i} \; \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \\
&= \left[\frac{m}{\Sigma_{j=1}^{n}\; \alpha_j} \right]^{\Sigma_{k=1}^{n}\; \alpha_k} \; \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}
\end{aligned} \tag{34}
$$

For the dual problem the indirect objective function is

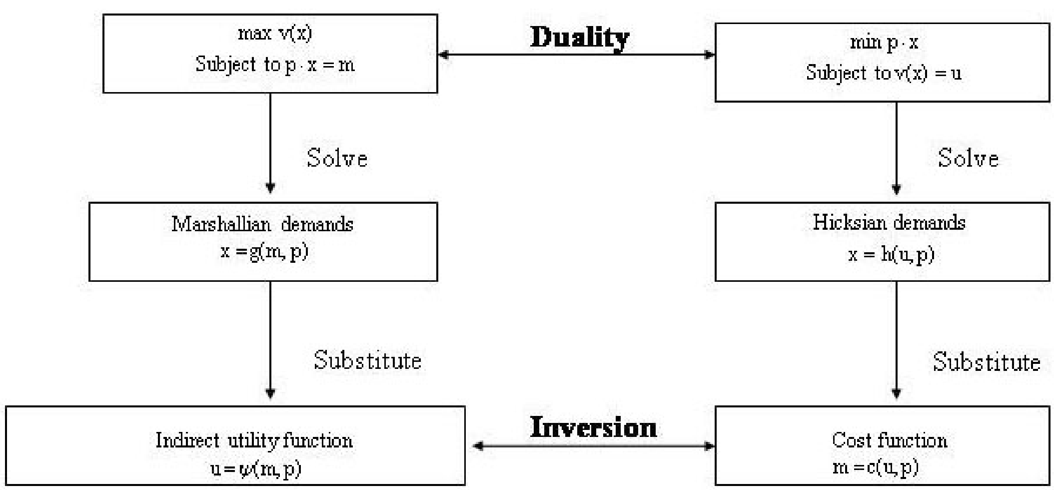
$$
m \;=\; \Sigma_{j=1}^{n}\, p_j\, h_j u,\, p\,) \;=\; c(u,\, p) \tag{35}
$$

This is called the cost (expenditure) function and specifies cost or expenditure as a function of prices and utility. We can also write it as follows

$$
c(u,\, p) \;=\; \min_{x}\, [p\, x\,:\; v(x) \;=\; u] \tag{36}
$$

3.4. **Inversion of ψ(m, p) and c(u, p).** Because c(u, p) = m, we can rearrange or invert it to obtain u as a function of m and p. This will give ψ(m, p). Similarly inversion of ψ(m ,p) will give c(u, p). These relationships between the utility maximization cost minimization problems are summarized in figure 1

FIGURE 1. Utility Maximization and Cost Minimization



3.5. **Properties of the cost function.** The cost function in the consumer problem has a number of analogous to its properties in the production problem.

3.5.1. *C.1.* The cost function is nondecreasing in p, increasing in u, and increasing in at least one p.

Let $p^1 \geq p^2$. Let x^1 be the cost minimizing input bundle with p^1 and x^2 be the cost minimizing input bundle with p^2. Then $p^2x^2 \leq p^2x^1$ because x^1 is not cost minimizing with prices p^2. Now $p^1x^1 \geq p^2x^1$ because $p^1 \geq p^2$ by assumption so that

$$C(p^1, y) = p^1\, x^1 \geq p^2\, x^1 \geq p^2\, x^2 \; = \; C(p^2, y)$$

Nonsatiation guarantees that the function will be increasing in u. Let $V(u_0)$ be the set of all bundles that are equivalent to or preferred to bundles that provide utility level u_0. Now let u_1 ¿ u_2. Because $V(u_1)$ is a subset of $V(u_2)$ if $u_1 \geq u_2$ then

$$C(u_1, p) = \min_x \{px : x \in V(u_1)\} \geq \min_x \{px : x \in V(u_2)\} = c(u_2, p)$$

The point is that if we have a smaller set of possible x's to choose from then cost must increase.

3.5.2. *C.2.* Positively linearly homogenous in p

$$C(u,\, \theta p) \; = \; \lambda\, C(u,\, p), \quad p > 0.$$

Let the cost minimization problem with prices p be given by

$$C(u,\; p) \; = \; \min_x \, \{px : x \epsilon V(u)\}, \quad u \in \mathcal{D}om\, V, p > 0, \tag{37}$$

where

$$\mathcal{D}om\, V \; = \; \{u \in R^1_+ : V(u) \neq \emptyset\}$$

The x vector that solves this problem will be a function of u and p, and is usually denoted h(u,p). This is the is Hicksian demand function. The cost function is then given by

$$C(u,\, p) \; = \; p\, h(u,\, p) \tag{38}$$

Now consider the problem with prices tp (p >0)

$$\begin{aligned} \hat{C}(y,\; tp) \; &= \; \min_x \, \{tpx : x \in V(u)\}, \quad u \in \mathcal{D}om\, V, p > 0 \\ &= t \, \min_x \, \{px : x \in V(y)\}, \quad y \in \mathcal{D}om\, V, p > 0 \end{aligned} \tag{39}$$

The x vector that solves this problem will be the same as the vector which solves the problem in equation 37, i.e., h(u,p). The cost function for the revised problem is then given by

$$\hat{C}(p,\, tp) \; = \; tp\, h(u,\, p) \; = \; tC(u, p) \tag{40}$$

3.5.3. *C.3.* C is concave and continuous in w

To demonstrate concavity let (p, x) and (p', x') be two cost-minimizing price-consumption combinations and let p"= tp + (1-t)p' for any $0 \leq t \leq 1$. Concavity implies that C(u, p") $\geq$ tC(u, p) + (1-t) C(u, p'). We can prove this as follows.

We have that C(u, p") = p"$\cdot$ x"= tp $\cdot$ x"+ (1-t)p' $\cdot$ x" where x" is the optimal choice of x at prices p" Because x" is not necessarily the cheapest way to obtain utility level u at prices p' or p,we have p $\cdot$ x"$\geq$ C(u, pw) and p'$\cdot$ x" $\geq$ C(u, p') so that by substitution C(u, p") $\geq$ tC(u, p) + (1-t) C(u, p'). The point is that if p $\cdot$ x" and p'$\cdot$ x" are each larger than the corresponding term is the linear combination then C(u, p") is larger than the linear combination.

Rockafellar [11, p. 82] shows that a concave function defined on an open set (p $>$ 0) is continuous.

3.6. **Shephard's Lemma.**

3.6.1. *Definition.* If indifference curves are convex, the cost minimizing point is unique. Then we have

$$\frac{\partial C(u,\ p)}{\partial p_i} \ =\ h_i\,(u,\ p) \tag{41}$$

which is a Hicksian Demand Curve

3.6.2. *Constructive proof using the envelope theorem.* The cost minimization problem is given by

$$C(y,w) \ =\ \min_x \ px : v(x) \ -\ u \ =\ 0 \tag{42}$$

The associated Lagrangian is given by

$$\mathfrak{L} \ = px - \lambda(v(x) \ -\ u) \tag{43}$$

The first order conditions are as follows

$$\frac{\partial \mathfrak{L}}{\partial x_i} \ =\ p_i \ -\ \lambda\,\frac{\partial v}{\partial x_i} \ = 0, \quad i \ =\ 1,\ \ldots,\ n \tag{44a}$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} \ =\ -\,(vx \ -\ u) \ = 0 \tag{44b}$$

Solving for the optimal x's yields

$$x_i(u,\ p) \ =\ h_i(u,\ p) \tag{45}$$

with C(u,p) given by

$$C(u,p) = px(u,p) = ph(u,p) \tag{46}$$

If we now differentiate 46 with respect to p_i we obtain

$$\frac{\partial C}{\partial p_i} \ =\ \Sigma_{j=1}^n\, p_j\,\frac{\partial x_j(u,p)}{\partial p_i} \ +\ x_i\,(u,p) \tag{47}$$

From the first order conditions in equation 44a (assuming that the constraint is satisfied as an equality) we have

$$p_j \ =\ \lambda\,\frac{\partial v}{\partial x_j} \tag{48}$$

Substitute the expression for p_j from equation 48 into equation 47 to obtain

$$\frac{\partial C}{\partial p_i} \;=\; \Sigma_{j=1}^{n} \,\lambda\, \frac{\partial v(x)}{\partial x_j}\, \frac{\partial x_j\,(u,p)}{\partial p_i} \;+\; x_i\,(u,p) \tag{49}$$

If $\lambda > 0$ then equation 44b implies[v(x)-u] = 0. Now differentiate equation 44b with respect to p_i to obtain

$$\Sigma_{j=1}^{n} \frac{\partial v(x(u,p))}{\partial x_j}\, \frac{\partial x_j\,(u,p)}{\partial p_i} \;=\; 0 \tag{50}$$

which implies that the first term in equation 49 is equal to zero and that

$$\frac{\partial C(u,p)}{\partial p_i} \;=\; x_i\,(u,p) \tag{51}$$

3.7. **The cost function and Marshallian demand functions.** If we substitute the indirect utility function in the Hicksian demand functions obtained via Shephard's lemma in equation 41, we get x in terms of m and p. Specifically

$$x_i \;=\; x_i(u,\,p) \;=\; h_i(u,\,p) \;=\; h_i[\psi(m,\,p),\,p] \;=\; g_i(m,\,p) \;=\; x_i(m,\,p) \tag{52}$$

3.8. **Cobb-Douglas Example.** The utility function is given by

$$v(x_1, x_2, \ldots, x_n) \;=\; \prod_{i=1}^{n} x_i^{\alpha_i} \;=\; x_1^{\alpha_1}\, x_2^{\alpha_2}\, x_3^{\alpha_3}\,\cdots \tag{53}$$

First set up the Lagrangian problem

$$\mathfrak{L} \;=\; \Sigma_{i=1}^{n}\, p_i\, x_i \;-\; \lambda\, (\prod_{i=1}^{n} x_i^{\alpha_i} \;-\; u\,) \tag{54}$$

The first order conditions are as follows

$$\frac{\partial \mathfrak{L}}{\partial x_i} \;= p_i \;-\; \lambda\, \left[\alpha_i\, x_1^{\alpha_1}\, x_2^{\alpha_2}\,\cdots\, x_{i-1}^{\alpha_{i-1}}\, x_i^{\alpha_i - 1}\, x_{i+1}^{\alpha_{i+1}}\,\cdots\,\right] \;=\; 0, \quad i \;=\; 1, \ldots, n \tag{55a}$$

$$= p_i \;-\; \frac{\alpha_i\, v}{x_i}\,\lambda \;=\; 0, \quad i \;=\; 1, \ldots, n \tag{55b}$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} \;=\; -\,\prod_{i=1}^{n} x_i^{\alpha} \;+\; u \;=\; 0 \tag{55c}$$

$$\tag{55d}$$

Taking the ratio of the ith and jth equations we obtain

$$\frac{p_i}{p_j} \;=\; \frac{\alpha_i\, x_j}{\alpha_j\, x_i} \tag{56}$$

We can now solve the equation for the jth quantity as a function of the ith quantity and the ith and jth prices. Doing so we obtain

$$\begin{aligned} x_j \;&= \frac{\alpha_j\, x_i\, p_i}{\alpha_i\, p_j} \\ &= \frac{\alpha_j\, x_1\, p_1}{\alpha_1\, p_j} \end{aligned} \tag{57}$$

where we treat the first good asymmetrically and solve for each demand for a good as a function of the first. Now substituting in the utility function we obtain

$$\begin{aligned} v &= \prod_{j=1}^{n} x_j^{\alpha_j} \\ &= \prod_{j=1}^{n} \left(\frac{\alpha_j \; x_1 \; p_1}{\alpha_1 \; p_j} \right)^{\alpha_j} \end{aligned} \tag{58}$$

Because x_1, p_1 and α_1 do not change with j, they can be factored out of the product to obtain

$$u = \left(\frac{x_1 \; p_1}{\alpha_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \tag{59}$$

We then solve this expression for x_1 as a function of u and the other x's. To do so we divide both sides by the product term to obtain

$$x_1^{\Sigma_{j=1}^{n} \alpha_j} \left(\frac{p_1}{\alpha_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} = \frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \tag{60}$$

We now multiply both sides by $\left(\frac{\alpha_1}{p_1} \right)^{\Sigma_{j=1}^{n} \alpha_j}$ to obtain

$$x_1^{\Sigma_{j=1}^{n} \alpha_j} = \frac{\left(\frac{\alpha_1}{p_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \tag{61}$$

If we now raise both sides to the power $\frac{1}{\Sigma_{j=1}^{n} \alpha_j}$ we find the value of x_1

$$x_1 = \left(\frac{\alpha_1}{p_1} \right) \left(\frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \tag{62}$$

Similarly for the other x_k so that we have

$$x_k = \left(\frac{\alpha_k}{p_k} \right) \left(\frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \tag{63}$$

Now if we substitute for the ith x in the cost expression we obtain

$$C = \Sigma_{i=1}^{n} p_i \left(\frac{\alpha_i}{p_i}\right) \left(\frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}}\right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}}$$

$$= \left(\Sigma_{i=1}^{n} \alpha_i\right) \left(\frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}}\right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \tag{64}$$

$$= \left(\Sigma_{i=1}^{n} \alpha_i\right) u^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}}$$

3.9. **The indirect utility function and Hicksian demands.** If we substitute C(u,p) in the Marshallian demands, we get the Hicksian demand functions

$$x_i = x_i(m, p) = g_i(m, p) = g_i[C(u,p), p] = h_i(u,p) = x_i(u, p) \tag{65}$$

3.10. **Roy's identity.** We can also rewrite Shephard's lemma in a different way. First write the identity

$$\psi\,(C(u,p), p] = u \tag{66}$$

Then totally differentiate both sides of equation 66with respect to p $_i$ holding u constant as follows

$$\frac{\partial\,\psi[C(u,p),\ p]}{\partial m}\,\frac{\partial\,C(u,p)}{\partial\,p_i} + \frac{\partial\,\psi[C(u,p),p]}{\partial\,p_i} = 0 \tag{67}$$

Rearranging we obtain

$$\frac{\partial\,C(u,p)}{\partial\,p_i} = \frac{-\frac{\partial\,\psi\,[C(u,p),p]}{\partial\,p_i}}{\frac{\partial\,\psi\,[C(u,p),p]}{\partial\,m}} = g_i(m,p) \tag{68}$$

where the last equality follows because we are evaluating the indirect utility function at income level m. Figure 2 makes these relationships clear.

4. Properties of indirect utility function

The indirect utility function has the following properties.

4.1. ψ**.1.** ψ(m,p) is nonincreasing in p, that is if p' $\geq$ p, ψ(m,p') $\leq$ ψ(m,p).

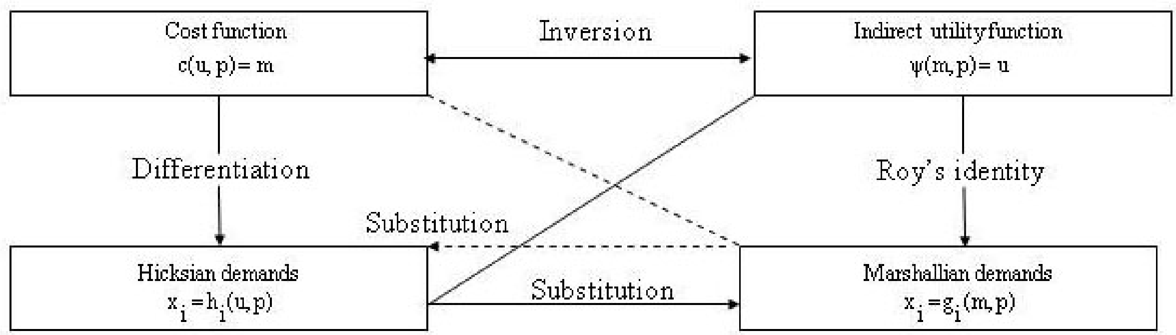
4.2. ψ**.2.** ψ(m,p) is nondecreasing in m, that is if m' $\geq$ m, ψ(m',p) $\geq$ ψ(m,p).

4.3. ψ**.3.** ψ(m,p) is homogeneous of degree 0 in (p, m) so that ψ(tm,tp) = ψ(m,p) for t $>$ 0.

4.4. ψ**.4.** ψ(m,p) is quasiconvex in p; that is $\{$p: ψ(m,p) $< \alpha\}$ is a convex set for all α.

4.5. ψ**.5.** ψ(m,p) is continuous for all p $>$ 0, m $>$ 0.

FIGURE 2. Demand, Cost and Indirect Utility Functions



5. DISCUSSION AND PROOFS OF PROPERTIES OF INDIRECT UTILITY FUNCTION

5.1. ψ**.1.** ψ(m,p) is nonincreasing in p, that is if p' $\geq$ p, ψ(m,p') $\leq$ ψ(m,p).

If prices go up, indirect utility cannot increase. Let B = {x: px <m} and B' = {x: p'x <m} for p' > p. Then B' is contained in B. Therefore the maximum of v(x) over B is at least as great as the maximum of v(x) over B'

5.2. ψ**.2.** ψ(m,p) is nondecreasing in m, that is if m' $\geq$ m, ψ(m',p) $\geq$ ψ(m,p).

If income goes up, indirect utility cannot decrease. Let B = {x: px <m} and B' = {x: px <m'} for m' > m. Then B is contained in B'. Therefore the maximum of v(x) over B' is at least as great as the maximum of v(x) over B.

5.3. ψ**.3.** ψ(m,p) is homogeneous of degree 0 in (p, m) so that ψ(tm,tp) = ψ(m,p) for t > 0.

This is called the absence of money illusion. If prices and income are multiplied by the same positive number, the budget set does not change. Specifically,

$$\sum_{i=1}^{n} (tp_i)\, x_i \; = tm$$

$$\Rightarrow \sum_{i=1}^{n} p_i\, x_i \; = m$$

With the same budget set, the utility maximization problem has the same solution.

5.4. ψ**.4.** ψ(m,p) is quasiconvex in p; that is {p: ψ(m,p) < α } is a convex set for all α.

Suppose p and p' are such that ψ(m,p) $\leq$ α, ψ(m,p') $\leq$ α. Let p" = tp + (1-t)t'. We want to show that ψ(m,p") $\leq$ α. Define the budget sets:

$$B \;= \{x : px < m\}$$
$$B' \;= \{x : p'x < m\}$$
$$B'' \;= \{x : p'x < m\}$$

We can show that any x in B' must be in either B or B'; that is that B ∪ B' ⊃ B". Assume not; then x is such that tpx + (1-t)p'x $|leq$ m, but Px > m and p'x >m. These two inequalities can be written as

$$\begin{aligned} tpx \; &> tm \\ (1-t)p'x \; &> (1-t)m \end{aligned} \tag{69}$$

Summing the two expressions in equation 69 we obtain

$$tpx \; + \; (1-t)p'x \; > \; m$$

But this contradicts the original assumption that x is in neither B or B'.

Now note that

$$\begin{aligned} \psi(m, p'') \; &= \max_x \, v(x), \quad \text{such that } x \in B'' \\ &\leq \max_x \, v(x), \quad \text{such that } x \in (B \cup B'); \text{because} \quad B \cup B' \supset B'' \\ &\leq \alpha \quad \text{because } \psi(m,p) \leq \alpha \text{ and } \psi(m,p') \leq \alpha \end{aligned}$$

5.5. ψ**.5.** ψ(m,p) is continuous for all p > 0, m > 0.

By the theorem of the maximum (given below) ψ(m,p) is continuous for p > 0, m > 0. In the utility maximization problem, f(x,λ) in the theorem of the maximum is the utility function. It does not depend on λ. The constraint set is those values of x that are in the budget set as parameterized by p and m. So λ in this case is (p,m). The indirect utility function ψ(m,p) is M(λ) while the ordinary demand functions x(m,p) are m(λ). The utility function is continuous by assumption. The constraint set is closed. If p > 0 and m > 0, the constraint set will be bounded. If some price were zero, the consumer might want to consume infinite amounts of this good. We rule that out.

Theorem 1 (Theorem of the Maximum)**.** *Let f(x,λ) be a continuous function with a compact range and suppose that the constraint set $\gamma(\lambda)$is a non-empty, compact-valued, continuous correspondence of λ. Then*

(i) *The function M(λ) =* $\max_x \{f(x,\lambda) : x \in \gamma(\lambda)\}$ *is continuous*
(ii) *The correspondence m(λ) =* $\{x \in \gamma(\lambda) : f(x,\lambda) = M(\lambda)\}$ *is nonempty, compact valued and upper semi-continuous.*

Proof: *See Berge* [1, p. 116].

6. Money Metric Utility Functions

6.1. **Definition of m(p,x).** Assume that the consumption set X is closed, convex, and bounded from below. The common assumption that the consumption set is X = R^L_+ = {x ∈ R^L: $x_\ell \geq 0$ for ℓ = 1, ... , L} is more than sufficient for this purpose. Assume that the preference ordering satisfies the properties given in section 1.8. Then for all x ∈ X, let BT(x) = {y∈BT | y ⪰ x}. For the price vector p, the money metric m(p,x) is defined by

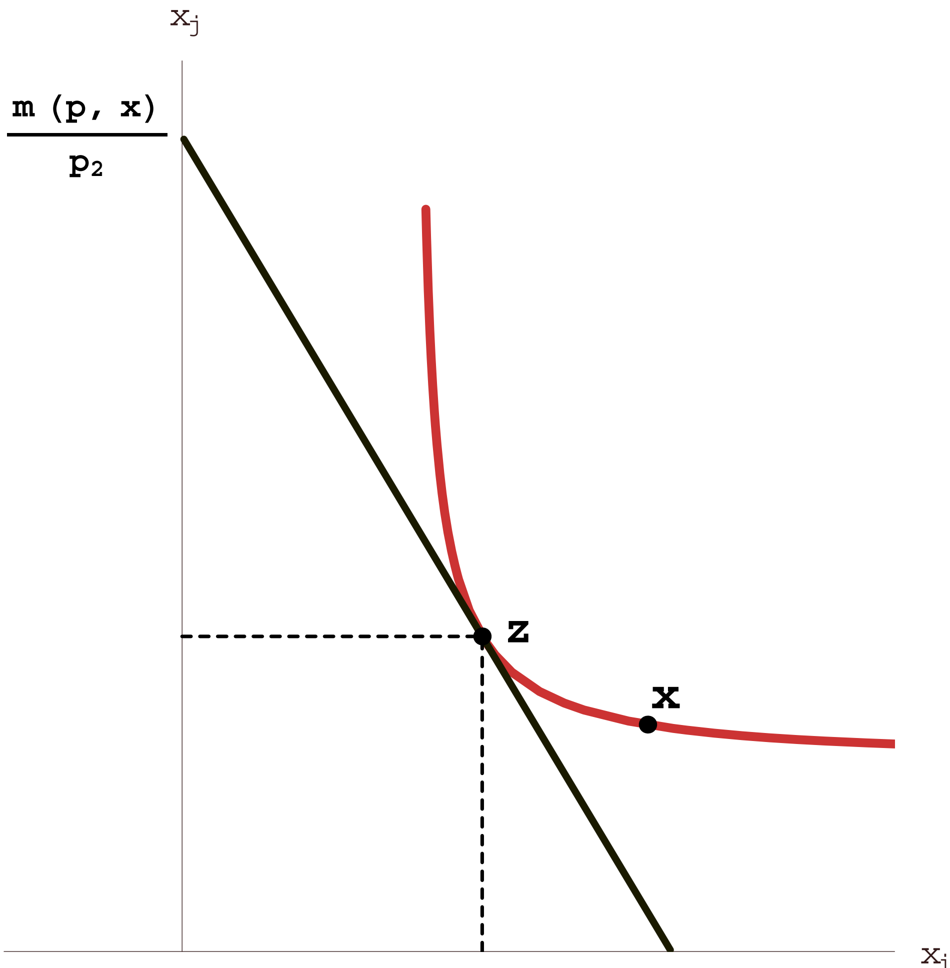
$$\begin{aligned} m(p,x) \; &= \min_{y \geq 0} \; py \\ s.t. \; y \; &\in \; BT(x) \end{aligned} \tag{70}$$

If p is strictly greater than zero and if x is a unique element of the least cost commodity bundles at prices p, then m(p,x) can be viewed as a utility function for a fixed set of prices. It can also be defined as follows.

$$m(p,x) \;=\; C(u(x),p) \tag{71}$$

The money metric defines the minimum cost of buying bundles as least as good as x. Consider figure 3

FIGURE 3. Utility Maximization and Cost Minimization



All points on the indifference curve passing through x will be assigned the same level of m(p,x), and all points on higher indifference curves will be assigned a higher level.

6.2. **Example of a money metric utility function.** Consider the Cobb-Douglas utility function

$$v(x_1, x_2, \ldots, x_n) \;=\; A \prod_{i=1}^{n} x_i^{\alpha_i} \;=\; A\, x_1^{\alpha_1}\, x_2^{\alpha_2}\, x_3^{\alpha_3} \cdots \tag{72}$$

The cost function associated with this utility function is given in equation 64, which we repeat here.

$$
\begin{aligned}
C &= (\,\Sigma_{i=1}^{n}\,\alpha_i\,) \left(\frac{v}{A\,\prod_{j=1}^{n}\,\left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}} \right)^{\frac{1}{\Sigma_{j=1}^{n}\alpha_j}} \\
&= (\,\Sigma_{i=1}^{n}\,\alpha_i\,)\; v^{\frac{1}{\Sigma_{j=1}^{n}\,\alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n}\alpha_j}}
\end{aligned} \tag{73}
$$

To obtain the money metric we replace v in equation 73 with v(x) from equation 72.

$$
\begin{aligned}
C &= (\,\Sigma_{i=1}^{n}\,\alpha_i\,)\; v^{\frac{1}{\Sigma_{j=1}^{n}\,\alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n}\alpha_j}} \\
\Rightarrow m(p,x) &= (\,\Sigma_{i=1}^{n}\,\alpha_i\,) \left(A \prod_{i=1}^{n} x_i^{\alpha_i} \right)^{\frac{1}{\Sigma_{j=1}^{n}\,\alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n}\alpha_j}} \\
&= (\,\Sigma_{i=1}^{n}\,\alpha_i\,) \left(A \prod_{i=1}^{n} \left(\frac{x_i\,p_i}{\alpha_i} \right)^{\alpha_i} \right)^{\frac{1}{\Sigma_{j=1}^{n}\,\alpha_j}}
\end{aligned} \tag{74}
$$

7. Money Metric Indirect Utility Functions

7.1. **Definition of μ(p,p^0,m).** For the price vectors p and p^0 and income m, the money metric indirect utility function is defined by

$$
\mu(p, p^0, m) \;=\; c(\psi(p^0, m), p) \tag{75}
$$

The indirect money metric utilty function defines the minimum cost of buying bundles at prices p that yield uility at least as large as than obtained when prices are p^0 and income is m. The money metric indirect utility function is sometimes called the indirect compensation function.

7.2. **Example of an indirect money metric utility function.** Consider the Cobb-Douglas utility function from equation 72

$$
v(x_1, x_2, \ldots, x_n) \;=\; \prod_{i=1}^{n} x_i^{\alpha_i} \;=\; x_1^{\alpha_1}\, x_2^{\alpha_2}\, x_3^{\alpha_3}\, \cdots
$$

The cost function associated with this utility function is given in equation 64, which we repeat here.

$$
C \;=\; (\,\Sigma_{i=1}^{n}\,\alpha_i\,)\; v^{\frac{1}{\Sigma_{j=1}^{n}\,\alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n}\alpha_j}} \tag{76}
$$

To obtain the money metric we replace v in equation 76 with ψ(p,m) from equation 34 which we also repeat here.

$$
\psi(p,\, m) \;=\; \left[\frac{m}{\Sigma_{j=1}^{n}\;\alpha_j} \right]^{\Sigma_{k=1}^{n}\;\alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \tag{77}
$$

Making the substitution we obtain

$$
\begin{aligned}
C &= \left(\Sigma_{i=1}^{n} \alpha_i \right) v^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \\
\Rightarrow m(p,x) &= \left(\Sigma_{i=1}^{n} \alpha_i \right) \left(\left[\frac{m}{\Sigma_{j=1}^{n} \ \alpha_j} \right]^{\Sigma_{k=1}^{n} \ \alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i^0} \right]^{\alpha_i} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \\
&= \left(\Sigma_{j=1}^{n} \alpha_j \right) \left[\frac{m}{\Sigma_{j=1}^{n} \ \alpha_j} \right] \prod_{j=1}^{n} \left(\left[\frac{\alpha_j}{p_j^0} \right]^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \\
&= m \prod_{j=1}^{n} \left(\left[\frac{\alpha_j}{\alpha_j} \right]^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \left(\prod_{j=1}^{n} \left(\frac{p_j}{p_j^0} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \\
&= m \left(\prod_{j=1}^{n} \left(\frac{p_j}{p_j^0} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}}
\end{aligned}
\tag{78}
$$

8. Properties of demand functions

Demand functions have the following properties

8.1. **Adding up or Walras law.**

$$\sum_{i=1}^{n} p_i h_(u, \ p) \ = \ \sum_{i=1}^{n} p_i \, g_i(m, \ p) \ = \ m \tag{79}$$

8.2. **Homogeneity.**

$$h_i \, (u, \ \theta \, p \,) \ = \ h_i \, (u, \ p \,) \ = \ g_i \, (\theta \, m, \ \theta \, p \,) \ = \ g_i \, (m, \ p) \tag{80}$$

The Hicksian demands are derivatives of a function that is homogeneous of degree one, so they are homogeneous of degree zero. Euler's theorem then implies that

$$\sum_{j=1}^{n} \frac{\partial \, h_j \, (u, \ p)}{\partial \, p_j} \, p_j \ = \ 0 \tag{81}$$

If all prices and income are multiplied by a constant t > 0, the budget set does not change and so the optimal levels of x(m,p) do not change. We can also write this in differential form using the Euler equation.

$$
\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial \, g_j(m, \ p)}{\partial \, p_j} \, p_j \ + \ \frac{\partial \, g_j(m, \ p)}{\partial \, m} \, m \ = 0 \\
&\Rightarrow \sum_{j=1}^{n} \frac{\partial \, g_j(m, \ p)}{\partial \, p_j} \, p_j \ = \frac{-\partial \, g_j(m, \ p)}{\partial \, m} \, m
\end{aligned}
\tag{82}
$$

8.3. **Symmetry.** The cross price derivatives of the Hicksian demands are symmetric, that is, for all i $\neq$ j

$$\frac{\partial h_j(u,\ p)}{\partial p_i} \ = \ \frac{\partial h_i(u,\ p)}{\partial p_j} \tag{83}$$

This is clear from the definition of the Hicksian demands as derivatives of the cost function. Specifically,

$$\frac{\partial C(u,\ p)}{\partial p_i} \ = \ h_i(u,\ p) \tag{84}$$

so that

$$\frac{\partial^2 C(u,p)}{\partial p_j\, \partial p_i} \ = \ \frac{\partial h_i(u,p)}{\partial p_j} \text{ and } \frac{\partial^2 C(u,p)}{\partial p_i\, \partial p_j} \ = \ \frac{\partial h_j(u,p)}{\partial p_i} \tag{85}$$

by Young's theorem on the equality of cross-partials.

8.4. **Negativity.** The nxn matrix formed by the elements $\frac{\partial h_i(u,p)}{\partial p_j}$ is negative semi-definite, that is, for any vector z, the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_i\, z_j \frac{\partial h_i(u,p)}{\partial p_j} \ \leq \ 0 \tag{86}$$

If the vector z is proportional to p, then the inequality becomes an equality and the quadratic form is zero. This means the matrix is negative semi-definite. This follows from the concavity of the cost function. If we denote $\frac{\partial h_i(u,p)}{\partial p_j}$ by s_{ij}, then we can write the entire matrix of cross partial derivatives as S = s_{ij}. This then implies that

$$z' S z \ \leq \ 0 \tag{87}$$

By the properties of a negative semi-definite matrix, this means that $s_{ii} \leq 0$, or that the Hicksian demand functions have a slope which is non-positive. This follows from concavity of cost, and does not require convex indifference curves.

8.5. **The Slutsky equation.** If we differentiate equation 65 with respect to p_j and then substitute from Shephard's lemma for $\frac{\partial C(u,p)}{\partial p_j}$,, we obtain

$$\begin{aligned}
x_i \ &= x_i(m,\ p) \ = \ g_i[C(u,p), p] \ = \ h_i(u,p) \ = \ x_i(u,\ p) \\
\Rightarrow \ x_i(u,p) \ &= h_i(u,p) \ = \ g_i[C(u,p), p] \\
\Rightarrow \ \frac{\partial x_i(u,p)}{\partial p_j} \ &= s_{ij} \ = \ \frac{\partial h_i(u,p)}{\partial p_j} \ = \ \frac{\partial g_i(m,p)}{\partial m} \frac{\partial C(u,p)}{\partial p_j} \ + \ \frac{\partial g_i(m,p)}{\partial p_j} \\
&= \frac{\partial g_i(m,p)}{\partial m} x_j \ + \ \frac{\partial g_i(m,p)}{\partial p_j} \\
&= \frac{\partial x_i(m,p)}{\partial m} x_j \ + \ \frac{\partial x_i(m,p)}{\partial p_j}
\end{aligned} \tag{88}$$

The last term in equation 88 is the uncompensated derivative of x_i with respect to p_j. To compensate for this, an amount, x_i, times $\frac{\partial g_i}{\partial m}$ must be added on. We can also write equation 88 as follows

$$\frac{\partial x_i(m,p)}{\partial p_j} \ = \frac{\partial x_i(u,p)}{\partial p_j} \ - \ \frac{\partial x_i(m,p)}{\partial m} x_j(m,p) \tag{89}$$

The first term is called the substitution effect, the second term the income effect. Notice that

$$\left|\frac{\partial\, h_i(u,p)}{\partial\, p_i}\right| > \left|\frac{\partial\, x_i(m,p)}{\partial\, p_i}\right| \tag{90}$$

when the i^{th} good is normal.

9. Integrability

In section 8 we determined that a system of demand equations satisfies

(i) Walras Law
(ii) Homogeneity of degree zero in prices and income
(iii) Symmetry
(iv) Negativity

The integrability question has to do with whether a system of equations satisfy these four properties whether there exists a utility function from which this system can be derived. The answer to this question is yes as indicated in the following theorem.

Theorem 2. *A continuously differentiable function x which maps R^n_+ into the real line is the demand function generated by some increasing, quasiconcave utility function if (and only if, when utility continuous, strictly increasing, and strictly quasiconcave) it satisfies Walras law, symmetry and negativity.*

This question is typically posed in terms of the cost function. One shows that a system of Hicksian demand functions satisfying Walras law, symmetry and negativity has an associated cost function from which it can be derived. Once one has the cost function, it can be used to obtain the utility function. Shephard's lemma (equation 41) states that

$$\frac{\partial C(u,\ p)}{\partial p_i} \;=\; h_i(u,\ p) \tag{91}$$

Ordinary (Marshallian demand equations can be written in terms of Hicksian demand equations (see equation 65) as follows

$$x_i(m,\ p) \;=\; x_i(C(u,p),\, p) \;=\; h_i(u,p) \;=\; x_i(u,\ p) \tag{92}$$

Combining equations 91 and 92 we can then write

$$\frac{\partial C(u,\ p)}{\partial p_i} \;=\; h_i(u,p) \;=\; x_i(u,\ p) = \; x_i(C(u,p),\, p), \quad i \;=\; 1,2,\ldots,k \tag{93}$$

Now suppose we have a system of demand functions x_i(m, p) for i = 1, 2, . . . , k. Now pick some point $x^0(m,p^0)$ and assign it an arbitrary level of utility u^0. Also assume that

$$C(u^0,\, p^0) \;=\; p^0 x(m^0, p^0) \tag{94}$$

If a cost function which generated this system of demand functions exists, then it must satisfy the system of partial differential equations given by

$$\frac{\partial C(u^0,\ p)}{\partial p_i} \;=\; h_i(u^0,p) \;=\; x_i(C(u^0,p),\, p), \quad i \;=\; 1,2,\ldots,k \tag{95}$$

If this system of partial differential equations has a solution, then x(m,p) is the demand system generated the cost function C(u, p). In order to understand the conditions for such a system to have a solution, consider the derivative of equation 95 with respect to p_j.

$$\frac{\partial^2 C(u^0,\ p)}{\partial p_j\, \partial p_i} \;=\; \frac{\partial\, x_i(C(u^0, p),\, p)}{\partial\, C}\, \frac{\partial\, C(u^0, p)}{\partial\, p_j} \;+\; \frac{\partial\, x_i(C(u^0, p),\, p)}{\partial\, p_j} \tag{96}$$

Substituting for $\frac{\partial\, C(u^0, p)}{\partial\, p_j}$ using Shephard's lemma and writing m in place of C we obtain

$$\frac{\partial^2 C(u^0,\ p)}{\partial p_j\, \partial p_i} \;=\; \frac{\partial\, x_i(m,\, p)}{\partial\, m}\, x_j(m,\, p) \;+\; \frac{\partial\, x_i(m,\, p)}{\partial\, p_j} \tag{97}$$

Notice that the left hand side of equation 97 is symmetric in i and j by Young's theorem. But if the left hand side of 97 is symmetric the right hand side of 97 must also be symmetric. This implies that the right hand side of 97 being symmetric is a necessary condition for the system of partial differential equations in 95 to have a solution. Frobenius' theorem states that symmetry of the right hand side of 97 is also sufficient for 95 to have a solution. Then notice that the right hand side of 97 is the Slutsky matrix associated with the demand system x(m,p). So symmetry of the Slutsky matrix is necessary and sufficient for a function C(u, p) to exist, from which x(p,m) can be derived. The question remaining is whether the function C(u,p) which solves 95 is a proper cost function.

We therfore need to verify that the properties of the cost function stated in subsection 3.5 hold for the function which solves the system in equation 95. These properties are

1. Nondecreasing in p, increasing in u, and increasing in at least one p.
2. Positively linearly homogenous in p
3. Concave and continuous in w

C(u, p) as a solution to 95 will be nondecreasing in p because Shephard's lemma shows that the derivative of C(u, p) with respect to any price is Hicksian demand which in nonnegative. The other properties in items 1 and 3 can be similarly shown. C(u,p) will be concave if it has a Hessian matrix which is negative semi-definite. But the Hessian of C(u,p) is just the Slutsky matrix. So if a system of demand equations has a negative semi-definite Slutsky matrix, then the solution to the system partial differential equations in 95 will be concave.

A continuously differentiable function x which maps R^n_+ into the real line is the demand function The bottom line is that a system of demand functions that satisfies Walras law, symmetry and negativity is consistent with some increasing, quasiconcave utility function.

10. Some Notes on Functions, Correspondences, and Functional Structure

10.1. **Functions.** By a function we mean a rule that assigns to each element in a set X, a unique element {f(x)} in another set Y. Consider the set X = R^1 and Y = R^1 and the rule f(x) = 3x. For any real x, the function assigns a unique real number in the set Y. We often use the following notation for a function

$$f : X \to Y$$

The set X is called the domain of the function f. The set of values taken by f, that is, the set y ∈ Y: (∃x) [y = f(x)] is called the range of f. The range of f will generally be smaller than Y. Consider the case where X is the rational numbers and Y is the real numbers. The function f(x) = 3x will not cover all members of Y. A function whose range is all of Y is said to be **onto** Y. If A is a subset of X, the image under f of A is defined to be the set of elements in Y such that y = f(x) for some x in A. This is denoted as f[A] and formally given by

$$f[A] = \{y \in Y : (\exists x)\ [x \in A \text{ and } y = f(x)]\}$$

The function f is **onto** Y iff Y = f[X]. If B is a subset of Y, the inverse image f^{-1}[B] is the set of x in X for which f(x) is in B. Formally

$$f^{-1}[B] = \{x \in X : f(x) \in B\}$$

The function f is **onto** Y iff the inverse image of each nonempty set of Y is nonempty. Consider again the example where X is the rational numbers. There are elements of Y such that the is no element of X that could generate them under the function f(x) = 3x. A function f: X → Y is called **one-to-one** if f(x_1) = f(x_2) only when x_1 = x_2. Consider the function mapping the real line into the real line where f(x) = x^2. The function is not one-to-one since f(-3) = f(3). The function is also not onto since there is no real x for which f(x) = -25.

10.2. **Correspondences.** Let X and Y be two sets. If with each element of X we associate a subset Γ(X) of Y, we say that the correspondence x → Γ(X) is a mapping of X into Y. The set Γ(x) is called the image of x under the mapping Γ. If the set Γ(x) always consists of a single element, we say that Γ is a function. Consider as an example the case where X is R^1_+ and Y is R^1 Now consider the mapping Γ(x) = y ∈ R^1: y ≤ -√x. This is a correspondence from X to Y.

10.3. **Functional Structure.** Functional structure has to do with the amount of information we have about a given mapping. Considered in a different fashion, it is about the number of constraints that we know are imposed on a mapping. For example it we know that y = 3x for all elements x ϵ X, then we know everything there is to know about the mapping. Alternatively, the mapping is very constrained in that no other rule satisfies this mapping. Consider the case where y = ax, but a is not known. In this case we know that the mapping is linear, but we cannot say much more than that about it. Consider a function which maps R^2 into R^1. Specifically consider the function $y = a x_1^2 + b x_2^2$. We know that the function is quadratic and that it has no linear or constant terms. It is also clear that it is homogeneous of degree 2 in the vector x because f(3x) = 9f(x). Thus the function is not completely general.

Functional structure relates to the way in which a mapping is constructed and the way in which the elements of the domain enter the mapping. Consider for example a general mapping from R^3 to R^1 that has the following form: y = f(x_1, x_2, x_3) = g(x_1, x_2) + h(x_3). While this is not much information about the mapping, it rules out functions such as y = $3x_1x_2 + 5x_1x_3 + 8x_2x_3$.

Functional structure is typically represented in two ways. The earliest and perhaps most common is through the use of differential constraints on the function. These constraints then imply something about the parent function. The difficulty with this approach is that it requires differentiability. The other approach is to consider structure on the function directly as for example $f(x) = F(h(x))$ where h is required to be homogeneous in all the x's.

10.4. **Functional Structure and Functional Equations.** Functional structure is often related to the solution of functional equations. Functional equations are equations in which the unknown (or unknowns) are functions. Such functions can be multi-place in the sense of having more than one argument and can also deal with several variables. The number of places in the equations need not equal the number of variables. For example, the famous Cauchy equation $f(x+y) = f(x) + f(y)$ is a function with one place but two variables. A linear function such as $f(z) = cz$ would satisfy the equation. As another example consider the cost function for a multiple output firm. Consider the restriction that the cost of producing the vector of outputs (x+y) is the sum of the cost of producing either vector individually, i.e., $C(w,x+y) = C(w, x) + C(w, y)$. The determination of the form that all cost functions satisfying this restriction must take involves solving the functional equation.

11. AGGREGATION ACROSS GOODS

11.1. **Separability and Aggregation.** Separability is related to the ability to aggregate variables in economic analysis. For example, is it reasonable to aggregate two or more types of cold cereal together in analyzing the demand for food. Or can the hours of male and female workers be added together for an analysis of productivity. Separability is specifically concerned with how the rate of substitution between two goods or factors is affected by levels of other goods or factors. For example the rate of substitution between beef and pork may or may not be affected by the amount of tofu consumed. If this rate is not affected by tofu consumption, then some types of aggregation of beef and pork may be possible. While beef and pork may not be separable from tofu in consumption, they may be separable from shirts. Separability is particularly important for aggregate analysis where inputs tend to come in generic bundles such as labor, capital and materials, and goods tend to come in bundles such as housing, food, transportation, entertainment and so on.

11.2. **Differential definition of separability.** Consider a function f depending on n variables with f being is twice differentiable and

$$\frac{\partial f}{\partial x_i} > 0 \tag{98}$$

Then variables x_i and x_j are separable from x_k if and only if

$$\frac{\partial \left(\frac{\frac{\partial f(x)}{\partial x_i}}{\frac{\partial f(x)}{\partial x_j}} \right)}{\partial x_k} = 0 \; \forall \, x \in \Omega^n \tag{99}$$

$$\Omega^n = \{ \, (x_1 \, , \, x_2, \ldots, \, x_n) \; = \; x \, \epsilon \, R^n \, | \, x \; \geq \; 0^n \text{ and } x \; \neq \; 0^n \}$$

This says that the marginal rate of substitution between x_i and x_j does not depend on the level of x_k. This definition is due to Leontief [8] and independently Sono [10].

11.3. **Intuition of differential definition of separability.** As x_k changes, the indifference curve when projected into x_i, x_j space will have the same slope at

$$(\bar{x}_i, \, \bar{x}_j)$$

In figure 4, changing x_3 to $\hat{x}_3$ does not change the set of x_1 and x_2 such that $u(x_1, x_2, x_3) \geq u(\bar{x}_1, \bar{x}_2, x_3)$. In figure 5, changing x_3 to $\hat{x}_3$ changes the set of x_1 and x_2 such that $u(x_1, x_2, x_3) \geq u(\bar{x}_1, \bar{x}_2, x_3)$.

FIGURE 4. Separability