COMPETITIVE EQUILIBRIUM AND SOCIETAL WELFARE

1. General Competitive Equilibrium

1.1. **Economic system.** An economic system is a collection of consumers, firms, and products and the environment in which they make decisions. We summarize this information as follows.

1.1.1. *The economic environment.* The actions taken by any agent depend on the opportunities presented to that agent. These opportunities depend on the *economic environment* of the agent. This environment is determined (constrained) by:

- **1:** basic physical and biological properties of the world in which the agent lives,
- **2:** the man-made technologies available and in use,
- **3:** the actions of other agents,
- **4:** the institutional framework of the economic system, and
- **5:** other legal, social or moral limits on choice,
- **6:** uncertain or stochastic factors that influence other parts of the environment.

1.1.2. *Goods.* There are L goods in the economy numbered ℓ = 1, 2, . . . , L. We characterize these goods in three different ways.

- **a:** The goods as objects of choice by consumers. A product vector is a list of the amounts of the various products:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{bmatrix}$$

- **b:** The goods as inputs or outputs of firms. A production vector is a list of the amounts of the various products utilized or produced by firms. A negative entry in a production vector is considered to be an input.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{bmatrix}$$

- **c:** An initial endowment of products in the economy. This is given by a vector $\omega = (\omega_1, \omega_2, \ldots, \omega_L) \in$ R^L.

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_L \end{bmatrix}$$

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1.1.3. *Prices.* We will assume that all L products are traded in the market at dollar prices that are publicly quoted. The prices are represented by a price vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{bmatrix} \in R^L$$

We assume that all prices are strictly positive, i.e. $p_\ell >> 0$.

1.1.4. *Consumers.* There are I consumers numbered 1, 2, ..., I , each characterized by a consumption set $X^i \subset R^L$ and a preference relation $\succeq_i$defined on X^i. The preference relation satisfies the following properties

1 complete in that for all $x_1, x_2 \in X$, we have $x_1 \succeq x_2$ or $x_2 \succeq x_1$ (or both)
2 transitive in that $\forall\, x_1, x_2, x_3 \in X$, if $x_1 \succeq x_2$ and $x_2 \succeq x_3$ then $x_1 \succeq x_3$.
3 locally nonsatiated in that for every $x_1 \in X$ and every $\varepsilon > 0$, there is $x_2 \in X$ such that $||x_2 - x_1 \leq \varepsilon||$ and $x_2 \succeq x_1$.
4 continuous in that for any sequence of pairs

$$\{ (x_1^n,\, x_2^n) \}_{n=1}^{\infty}\; with\; x_1^n \succeq\; x_2^n \forall\, n,$$
$$x_1 \,=\, \lim_{n \to \infty} x_1^n,\; and\; x_2 \,=\, \lim_{n \to \infty} x_2^n,$$
we have $x_1 \succeq x_2$.

The preference relation for the i^{th} consumer can be represented by a utility function $v^i(x)$. The function $v^i(x)$ represents $\succeq$ in the sense that $x_1 \succeq x_2$ iff $v^i(x_1) \geq v^i(x_2)$.

1.1.5. *Firms.* There are J firms numbered 1, 2, ..., J , each characterized by a production technology $Y^j \in R^L$. A technology is a description of process by which inputs are converted in outputs. In this section we will differentiate between inputs and outputs by whether they enter the production technology as negative or positive elements. If we denote the negative elements of a given vector y by y^- and the positive elements by y^+,then the technology set for the j^{th} firm is given by

$$Y^j \;=\; \{(y^-,\, y^+) :\;\; (y^-,\, y^+)\, \epsilon\, R_+^L \,:\, y^- \text{ can produce } y^+ \}$$

The set consists of those combinations of (y^-, y^+) such that y^+can be produced from the given y^-. For example the vector {-2,5,-4,-1,0,6} implies that products one, three and four are inputs, products two and six are outputs, and product five is not used in this technology.

We assume that each firm chooses its production plan $y^j \in Y^j$ in order to maximize profits. Specifically, the firm solves the problem

$$\max_{y^j} \left[\sum_{\ell=1}^{L} p_\ell\, y_\ell^j \right] \; such\; that\; [y^j \;\in\; Y^j\,] \tag{1}$$

1.1.6. *Initial Endowments.* Each consumer has an initial endowment vector of products $\omega^i \in R^L$ where $\bar{\omega} = \sum_i^I \omega^i$. This societal endowment can either be consumed or used to produce other commodities. The i^{th} consumer has initial endowment ω_ℓ^i of product ℓ.

1.1.7. *Goods available for consumption.* The total amount of goods available for consumption in the economy is the sum of initial endowments and those produced by firms using the initial endowments as inputs.

$$\text{Net supply of good}_\ell \;=\; \omega_\ell + \sum_{j=1}^{J} y_\ell^j$$

where y_ℓ^j is the net production of product ℓ by the j^{th} firm.

1.1.8. *Distribution of Profits.* Each consumer has a claim to a share $\theta_j^i \in [0,1]$ of the profits of firm j where $\sum_{i=1}^{I} \theta_j^i = 1$ for every firm j. Thus if firm j chooses production plan y^j , the profit earned by firm j is $\pi^j = py^j$, and consumer i's share of this profit is given by $\theta_j^i(py^j)$. Consequently, consumer i's total wealth is given by $p\omega^i + \sum_{j=1}^{J} \theta_j^i \pi^j$. Note that this means that all wealth is either in the form of endowment or firm share; there is no longer any exogenous wealth w or income m. Of course, this depends on firms' decisions, but part of the idea of the equilibrium is that production, consumption, and prices will all be simultaneously determined.

1.1.9. *Competitive behavior.* All agents in the economy are assumed to be competitive.

1.2. **Allocations.** The possible outcomes in this economy are called allocations. An allocation ($x^1, x^2, \ldots, x^I, y^1, y^2, \ldots, y^J$) is a consumption vector $x^i \in X^i$ for each consumer i = 1, 2, . . . , I, and a production vector $y^j \in Y^j$ for each firm. An allocation is feasible if

$$\sum_{i=1}^{I} x_\ell^i \;\leq\; \omega_\ell \;+\; \sum_{j=1}^{J} y_\ell^j, \quad \ell \;=\; 1, 2, \ldots, L \tag{2}$$

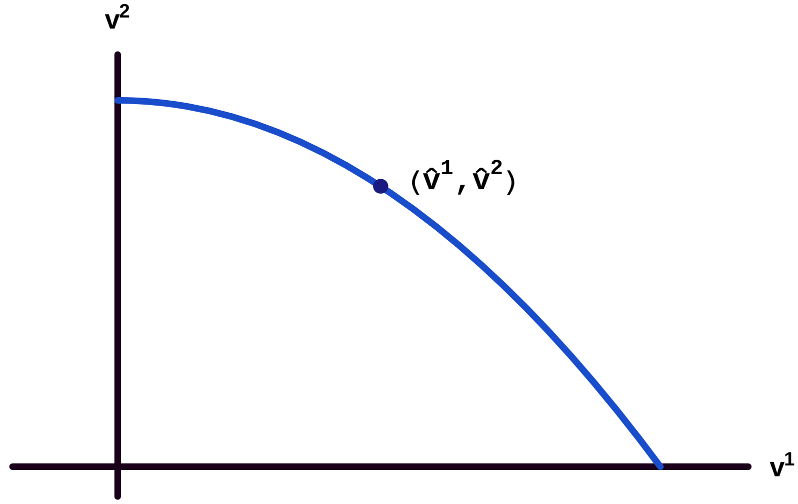
That is, if total consumption of each commodity is no larger than the total amount of that commodity available.

1.3. **Pareto Optimality.** One of the *goals* economists often state for an economy is efficiency. Another is to *make everyone better off*. The concept of "making everybody better off" is formalized by Pareto optimality. When an economist talks about efficiency, he is usually referring to situations where no one can be made better off without making some one else worse off. This is the notion of Pareto optimality.

Definition 1 (Pareto Optimality)**.** A feasible allocation ($x^1, x^2, \ldots, x^I, y^1, y^2, \ldots, y^J$) is *Pareto optimal* or Pareto efficient if there is no other feasible allocation ($x^{1\prime}, x^{2\prime}, \ldots, x^{I\prime}, y^{1\prime}, y^{2\prime}, \ldots, y^{J\prime}$) such that $u^i(x^{i\prime}) \geq u^i(x^i)$ for all i, with strict inequality for at least one i.

A Pareto optimal allocation is efficient in the sense that there is no other way to reorganize society's productive facilities in order to make somebody better of without harming somebody else. Notice that we don't seem care about producers in this definition of Pareto optimality. This is reasonable because a profit-maximizing firm will never buy inputs it doesn't use or produce output it doesn't sell,and given that all profits of firms end up in the hands of consumers who then spend them to buy goods, x^i. If we draw a utility possibility frontier in two dimensions, as in figure 1 Pareto optimal points are ones that lay on the northeast frontier. Pareto optimality doesn't say anything about equity. An allocation that gives one person everything and the other nothing may be Pareto optimal.

FIGURE 1. Utility Possibility Frontier



1.4. **Competitive Equilibrium.** There are three requirements for a competitive equilibrium, corresponding to the requirements that producers optimize, consumers optimize, and that ”markets clear” at the equilibrium prices. An equilibrium will then consist of a production plan y^j for each firm, a consumption vector x^i for each consumer, and a price vector p.

Definition 2 (Competitive Equilibrium)**.** The allocation $(x^{1*}, x^{2*}, \ldots, x^{I*}, y^{1*}, y^{2*}, \ldots, y^{J*})$ and price vector $p^* \in R^L$ constitutes a competitive or *Walrasian* equilibrium if the following conditions are satisfied

Profit Maximization: For every firm the set of inputs used and outputs produced maximize profit at those prices given the firms technology. Specifically, for each firm j, y^j* solves

$$\max_{y^j} \left[\sum_{\ell=1}^{L} p_\ell^* \, y_\ell^j \right] \text{ such that } [y^j \in Y^j] \tag{3}$$

Utility Maximization: For each consumer the consumption bundle is maximal for $\succeq_i$ in the budget set defined by the initial endowment (valued at the equilibrium prices) and their share of the profits of the J firms in the economy. Specifically, for each consumer i, x^i* solves

$$\begin{aligned} &\max_{x^i} v^i(x^i) \\ &\text{such that } \sum_{\ell=1}^{L} p_\ell^* \, x_\ell^i \le \sum_{\ell=1}^{L} p_\ell^* \, \omega_\ell^i + \sum_{j=1}^{J} \theta_j^i p_\ell^* \, y_\ell^{j*} \end{aligned} \tag{4}$$

Market Clearing: The total consumption of products by consumers is equal to initial endowments plus the net output of firms. Specifically, for each good $\ell = 1, 2, \ldots, L$,

$$\begin{aligned} \sum_{\imath=1}^{I} x_{\ell}^{i*} \;&\leq\; \sum_{i=1}^{I} \omega_{\ell}^{i} \;+\; \sum_{j=1}^{J} y_{\ell}^{j*} \\ \Rightarrow \sum_{\imath=1}^{I} x_{\ell}^{i*} \;&\leq\; \omega_{\ell} \;+\; \sum_{j=1}^{J} y_{\ell}^{j*} \end{aligned} \tag{5}$$

Note that we won't know the firm's profit or the value of the consumer's initial endowment until after the price vector is determined. But, if we don't know the value of the initial endowment or the firm's profit, we can't derive consumers' demand functions, and so we can't solve the utility maximization problem. But then that is the nature of general equilibrium problems.

Operationally, the requirements for an equilibrium can be written as:

1. For each consumer, $x^{i*}(p,\omega^i,\theta^i)$ solves the utility maximization problem. Add up the individual demand curves to get aggregate demand, D(p), as a function of prices. This is fairly straightforward if the individual demand curves take the Gorman form.
2. For each firm, $y^j(p)$ solves the profit maximization problem. Add up the individual supply curves to get aggregate supply, S (p), as a function of prices.
3. Find the price where $D(p^*) = S(p^*)$.

Because the demand curves $x^i(p)$ and supply curves $y^j(p)$ are homogeneous of degree zero in prices, we know that if p^* induces a competitive equilibrium, αp^* also induces a competitive equilibrium for any $\alpha > 0$. This allows us to normalize the prices without loss of generality. We often do so by setting the price of one of the goods equal to 1.

1.4.1. *Walras Law and Market Clearing.* In determining an equilibrium for the economy, it is useful to note that if the markets for L-1 of the good clear at price p^*, then the L^{th} market must clear as well, provided that consumers satisfy Walras Law and p^* is strictly positive. We can state this in the form of a lemma.

Lemma 1. *If the allocation (x^1, x^2, ..., x^I, y^1, y^2, ..., y^J) and price vecotr $p >> 0$ satisfy the market clearing condition 5 for all goods $\ell \neq k$, and if every consumer's budget constraintis satisfied with equality, so that $\sum_{\ell=1}^{L} p_\ell\, x_\ell^i \;=\; \sum_{\ell=1}^{L} p_\ell\, \omega_\ell^i \;+\; \sum_{j=1}^{J} \theta_j^i p_\ell\, y_\ell^j$ for all i, then the market for good k also clears. That is, if*

$$\sum_{i=1}^{I} x_{\ell}^{i}(p^*) \;=\; \sum_{i=1}^{I} \omega_{\ell}^{i} \;+\; \sum_{j=1}^{J} y_{\ell}^{j}(p^*) \quad \forall \ell \;\neq\; k \tag{6}$$

then

$$\sum_{i=1}^{I} x_{k}^{i}(p^*) \;=\; \sum_{i=1}^{I} \omega_{k}^{i} \;+\; \sum_{j=1}^{J} y_{k}^{j}(p^*) \tag{7}$$

Proof. Adding up the consumers' budget constraints over the I consumers and rearranging terms, we obtain

$$\sum_{\ell \neq k} p_\ell \left(\sum_{i=1}^{I} x_\ell^i \;-\; \sum_{i=1}^{I} \omega_\ell^i \;-\; \sum_{j=1}^{J} y_\ell^j \right) \;=\; -p_k \left(\sum_{i=1}^{I} x_k^i \;-\; \sum_{i=1}^{I} \omega_k^i \;-\; \sum_{j=1}^{J} y_k^j \right) \tag{8}$$

By market clearing in goods $\ell \neq$ k, the left hand side of equation 8 is equal to zero. Thus the right hand side must be zero as well. Becuase $p_k > 0$, this implies we have market clearing in good k.

□

This lemma is a direct consequence of the idea that total wealth must be preserved in the economy. The nice thing about it is that when you are only studying two markets, as is done in the partial equilibrium approach, if one market clears, the other must clear as well. Hence the study of two markets really reduces to the study of one market.

2. Partial Equilibrium Competitive Analysis

We now turn away from the general model to a simple case, known as partial equilibrium. Marshallian partial equilibrium analysis envisions the market for one good that constitutes a small part of the total economy, often a two commodity world. The small size of the market facilitates two important simplifications. First, when the expenditure on the good in question is a small portion of the consumer's total expenditure, only a small fraction of any additional dollar of wealth will be spent on this good. As a consequence, wealth effects for this good will be small. Remember that quasilinear preferences correspond to the case where there are no wealth effects in the non-numeraire good. Secondly, with substitution possibilities spread over ℓ -1 other commodities, the small size of the market under study should lead to minimal price effects in these ℓ -1 markets from changes in the market in question. Because of this *fixity* of other prices, we are justified in treating expenditure on all these other goods as a single composite commodity (section 2.1), which we call the numeraire commodity. In these partial equilibrium models we typically denote the amount of consumption of this *numeraire* commodity by the i^{th} consumer as m^i. So, basically what we do in a partial equilibrium approach is assume that there are two goods: a composite commodity (the numeraire m^i) whose price is set equal to 1, and the good of interest, x_1^i with price p_1.

2.1. **Composite Commodity Theorem.**

A composite commodity is a group of goods for which all prices move together.These goods can be treated as a single commodity. The individual behaves as if he or she is choosing between other goods and spending on this entire composite group.

Suppose that consumers choose among L goods but that we are specifically interested in one of them, say x_1. The demand for x_1 will depend on the prices of the other L-1 commodities. Assume that these prices all vary together (so that the relative prices of $\mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_L$ do not change). Let $\mathrm{p}_2^0, \mathrm{p}_3^0, \ldots, \mathrm{p}_L^0$ represent the initial prices of these other commodities. Define the composite commodity m to be total expenditures on $\mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_L$ at the initial prices, i.e.,

$$m \;=\; p_2^0\, x_2 \;+\; p_3^0\, x_3 \;+\; \ldots \;+\; p_L^0\, x_L \tag{9}$$

The individuals initial budget constraint is

$$\begin{aligned} w \;&= p_1\, x_1 \;+\; p_2^0\, x_2 \;+\; p_3^0\, x_3 \;+\; \ldots \;+\; p_L^0\, x_L \\ &= p_1\, x_1 \;+\; m \end{aligned} \tag{10}$$

where w is the initial wealth or total income. If we assume that all of the prices $\mathrm{p}_2, \mathrm{p}_3, \ldots, \mathrm{p}_L$ change by the same factor (t) then the budget constraint becomes

$$\begin{aligned} w \;&= p_1\, x_1 \;+\; tp_2^0\, x_2 \;+\; tp_3^0\, x_3 \;+\; \ldots \;+\; tp_L^0\, x_L \\ &= p_1\, x_1 \;+\; tm \end{aligned} \tag{11}$$

The factor of proportionality, t, plays the same role in the budget constaint of this person as does the price of x_2 in a two good world. Changes in p_1 or t induce substitution effects between x_1 and the other goods. As long as $\mathrm{p}_2, \mathrm{p}_3, \ldots, \mathrm{p}_L$ move together, we can confine our examination of demand to choices between buying x_1 and *everything else*. The theorem makes no prediction about how choices of $\mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_L$ behave, it only focuses on total spending on $\mathrm{x}_2, \mathrm{x}_3, \ldots, \mathrm{x}_L$.

2.2. **Formulation of the Quasilinear Partial Equilibrium Model.** There are two goods in the model, x_1 and m. Let x_1^i and m^i be consumer i's consumption of the commodity of interest and the numeraire commodity, respectively. Assume that each consumer i = 1, 2, . . . ,I has quasilinear utility of the form

$$v^i(m^i,\, x_1^i) \;=\; m^i \;+\; \phi^i(x_1^i) \tag{12}$$

The consumption set is $R \times R_+$. This allows for m to be negative. We normalize $v^i(0, 0) = \phi^i(0) = 0$. We assume that $\phi^i(\cdot)$ is bounded above and twice differentiable and ${\phi^i}'(x_1^i) > 0$ and ${\phi^i}''(x_1^i) < 0$ for all $x_1^i \geq 0$. That is, we assume that the consumer's utility is increasing in the consumption of x_1 and that her marginal utility of consumption is decreasing. AS we stated before, the composite commodity m^i has a price equal to 1 and x_1^i with price p_1. There are J firms in the economy. Each firm can transform m into x according to cost function

$$c^j(q^j) \tag{13}$$

where q^j is the quantity of x_1 that firm j produces, and $c^j(q^j)$ is the number of units of the numeraire commodity needed to produce q^j units of x_1. Because the price of m is 1, $c^j(q^j)$ tells us the number of units of m needed to produce x_1 and the cost of those units. Thus, letting z^j denote firm j's use of good m as an input, its technology set is therefore

$$Y^j \;=\; \{(-z^j,\, q^j) | q^j \;\geq\; 0 \text{ and } z^j \;\geq\; c^j(q^j)\} \tag{14}$$

That is, you have to spend enough of good m to produce q^j units of x. We assume that $c^j(q^j)$ is twice differentiable with ${c^j}'(q^j) > 0$ and ${c^j}''(q^j) < 0$.

We assume there is no initial endowment of x_1, but that consumer i has endowment of m equal to $\omega_m^i > 0$ of the product m. We assume that the total endowment of m is $\sum_{i=1}^{I} \omega_m^i = \omega_m$.

2.3. **An aside on optimization problems with inequality constraints.** Consider the problem to maximize f(x) subject to $g_i(x) \leq 0$ for i = 1, ... ,I_1, $g_i(x) \geq 0$ for i = I_1 + 1, . . . ,I, and $h_j(x) = 0$ for j = 1, . . . ,J, and $x \in X$, where X is an open set in R^n. The problem would be written as follows:

$$\begin{aligned} &\max_{x} f(x, a) \\ &\text{such that} \\ g_i(x, a) \;&\leq 0,\, i = 1,\, 2,\, \cdots,\, I_1 \\ g_i(x, a) \;&\geq 0,\, i \;= I_1 \,+\, 1,\, I_1 \,+\, 2,\, \cdots,\, I \\ h_j(x, a) &= 0,\; j \;=\; 1,\, 2,\, \cdots,\, J \end{aligned} \tag{15}$$

2.3.1. *Optimality conditions.* The necessary conditions for this problem are known as Karush-Kuhn-Tucker conditions and can be expressed as follows where $\bar{x}$ is a feasible solution:

$$\begin{aligned} \nabla f(\bar{x}) \;-\; \Sigma_{i=1}^{I}\, u_i \,\nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) \;&=\; 0 \\ u_i g_i(\bar{x}) \;&= 0, \quad for\ i \;=\; 1, \ldots, I \\ u_i \;&\geq 0, \quad for\ i \;=\; 1, \ldots, I_1 \\ u_i \;&\leq 0, \quad for\ i \;=\; I_1 + 1, \ldots, I \end{aligned} \tag{16}$$

In a maximization problem set up this way, the constraints expressed as less than will have positive multipliers.

2.3.2. *Problems where the constraints are non-negativity constraints on the variables.* Consider problems of the type: max f(x) subject to $g_i(x) \leq 0$ for i = 1, ... ,I_1, $g_i(x) \geq 0$ for i = I_1 + 1, . . . ,I, and $h_j(x) = 0$ for j = 1, . . . ,J, and $x \geq 0$. Such problems with nonnegativity restrictions on the variables frequently arise in practice in situations where prices or quantities are positive. Clearly, the KKT conditions discussed earlier would apply as usual. However, it is sometimes convenient to eliminate the Lagrangian multipliers associated with $x \geq 0$. To eliminate these multipliers explicitly add the constraint as $d_k(x) \geq 0$ for k = 1, . . . ,K, where $d_k(x) = x_\ell$ where ℓ is the elemet of the x vector that d_k specifies to be positive. The conditions are

$$\nabla f(\bar{x}) - \sum_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \sum_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) - \sum_{k=1}^{K} \delta_k \nabla d_k(\bar{x}) = 0 \tag{17a}$$

$$u_i g_i(\bar{x}) = 0, \quad for\ i = 1, \ldots, I \tag{17b}$$

$$\delta_k d_k(\bar{x}) = 0, \quad for\ k = 1, \ldots, K \tag{17c}$$

$$u_i \geq 0, \quad for\ i = 1, \ldots, I_1 \tag{17d}$$

$$u_i \leq 0, \quad for\ i = I_1 + 1, \ldots, I \tag{17e}$$

$$\delta_k \leq 0, \quad for\ k = 1, \ldots, K \tag{17f}$$

The derivative of $d_k(x)$ with respect to ℓ^{th} variable x_ℓ is 1, so the 17 reduces to

$$\begin{aligned} \nabla f(\bar{x}) - \sum_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \sum_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) - \sum_{k=1}^{K} \delta_k &= 0 \\ \Rightarrow \nabla f(\bar{x}) - \sum_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \sum_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) &= \sum_{k=1}^{K} \delta_k \end{aligned} \tag{18}$$

Given that $\delta_k \leq 0$, $\nabla f(\bar{x}) - \sum_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \sum_{j=1}^{J} \lambda_j \nabla h_j(\bar{x})$ must be less than zero. The conditions then reduce to

$$\begin{aligned} \nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x}) &\leq 0 \\ \left[\nabla f(\bar{x}) - \Sigma_{i=1}^{I} u_i \nabla g_i(\bar{x}) - \Sigma_{j=1}^{J} \lambda_j \nabla h_j(\bar{x})\right]' x &= 0 \\ u_i\ g_i(\bar{x}) &= 0, \quad for\ i = 1, \ldots, I \\ u_i &\geq 0, \quad for\ i = 1, \ldots, I_1 \\ u_i &\leq 0, \quad for\ i = I_1 + 1, \ldots, I \end{aligned} \tag{19}$$

2.4. **Analysis of the Quasilinear Partial Equilibrium Model.**

2.4.1. *Formal analysis of the model.* In order to find an equilibrium for this model, we need to derive the firms' supply functions, the consumers' demand functions, and find the market-clearing price.

Firm equilibrium: Given the equilibrium price p^* for good 1, firm j's equilibrium output q^{j*} must solve

$$\max_{q^j \geq 0} p_1^* q^j - c^j(q^j) \tag{20}$$

which has the necessary and sufficient first-order condition

$$p_1^* \leq \frac{d\,c^j(q^{j*})}{dq^j}, \text{with equality if } q^{j*} > 0 \tag{21}$$

Consumer equilibrium: Consumer i's equilibrium consumption vector (x_1^i,m^i) must solve

$$\begin{aligned} &\max_{x_1^i \in R_+,\, m^i \in R} \; m^i + \phi^i(x^i) \\ &\text{such that } m^i + p_1^* x_1^i \leq \omega_m^i + \sum_{j=1}^{J} \theta_j^i \left(p_1^* q^{j*} - c^j(q^{j*})\right) \end{aligned} \tag{22}$$

We know that the budget constraint must hold with equality. We can substitute it into the objective function for m^i so that the maximization problem can be written solely in terms of x_1 as follows

$$\max_{x_1^i \in R_+} \phi^i(x^i) - p_1^* x_1^i + \left[\omega_m^i + \sum_{j=1}^{J} \theta_j^i \left(p_1^* q^{j*} - c^j(q^{j*})\right)\right] \tag{23}$$

The first-order condition is

$$p_1^* \geq \frac{d\,\phi^i(x_1^{i*})}{dx_1^i}, \text{with equality if } x_1^{i*} > 0 \tag{24}$$

Market clearing: Remember that lemma 1 said if one of the markets clears we know that the other market must clear as well. Specifically, we will determine the levels of good 1 produced and consumed (x_1^{1*}, x_1^{2*}, $\ldots$, x_1^{I*}, q^{1*},q^{2*},$\ldots$,q^{J*}) with the understanding that consumer i's equilibrium consumption of m is

$$m^{i*} = \left[\omega_m^i + \sum_{j=1}^{J} \theta_j^i \left(p_1^* q^{j*} - c^j(q^{j*})\right)\right] - p_1^* x_1^i \tag{25}$$

and firm j's equilibrium usage of m as an input is given by

$$z^{j*} = c^j(q^{j*}) \tag{26}$$

Note that if we sum equation 25 over all consumers we obtain

$$
\begin{aligned}
\sum_{i=1}^{I} m^{i*} &= \sum_{i=1}^{I} \left[\omega_m^i + \sum_{j=1}^{J} \theta_j^i \left(p_1^* q^{j*} - c^j(q^{j*}) \right) \right] - \sum_{i=1}^{I} p_1^* x_1^i \\
&= \sum_{i=1}^{I} \omega_m^i + \sum_{i=1}^{I} \left[\sum_{j=1}^{J} \theta_j^i \left(p_1^* q^{j*} - c^j(q^{j*}) \right) \right] - p_1^* \sum_{i=1}^{I} x_1^i \\
&= \sum_{i=1}^{I} \omega_m^i + p_1^* \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_j^i q^{j*} - \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_j^i c^j(q^{j*}) - p_1^* \sum_{i=1}^{I} x_1^i \\
&= \sum_{i=1}^{I} \omega_m^i + p_1^* \sum_{j=1}^{J} q^{j*} \sum_{i=1}^{I} \theta_j^i - \sum_{j=1}^{J} c^j(q^{j*}) \sum_{i=1}^{I} \theta_j^i - p_1^* \sum_{i=1}^{I} x_1^i \\
&= \omega_m + p_1^* \sum_{j=1}^{J} q^{j*} - \sum_{j=1}^{J} c^j(q^{j*}) - p_1^* \sum_{i=1}^{I} x_1^i, \quad \sum_{i=1}^{I} \theta_j^i = 1, \\
&= \omega_m - \sum_{j=1}^{J} c^j(q^{j*}), \quad \sum_{j=1}^{J} q^{j*} = \sum_{i=1}^{I} x_1^i \quad \text{in equilibrium}
\end{aligned}
\tag{27}
$$

So the amount of the numeraire good consumed by all the consumers is $\omega_m - \sum_{j=1}^{J} c^j(q^{j*})$.

To find an equilibrium, we basically find a price vector such that aggregate demand for x_1 equals aggregate supply of q, $\sum_{i=1}^{I} x^i(p^*) = \sum_{j=1}^{J} q^j(p^*)$, i.e. the market for the consumption commodity x_1 clears. Then, we use the budget equation to compute the equilibrium level of m^i for each consumer using the lemma which us that the market for the numeraire must clear as well. The specific coniditions for an equilibrium in this economy are

Definition 3 (Equilibrium Conditions for Quasilinear Economy)**.**

$$p_1^* \geq \frac{d\,\phi^i(x_1^{i*})}{dx_1^i}, \text{ with equality if } x_1^{i*} > 0,\ i = 1, 2, \ldots, I \tag{28a}$$

$$p_1^* \leq \frac{d\,c^j(q^{j*})}{dq^j}, \text{ with equality if } q^{j*} > 0,\ j = 1, 2, \ldots, J \tag{28b}$$

$$\sum_{i=1}^{I} x_1^{i*} = \sum_{j=1}^{J} q^{j*} \tag{28c}$$

At any interior solution, equation 28a says that consumer i's marignal benefit from consuming an additional unit of good 1, $\frac{d\,\phi^i(x_1^{i*})}{dx_1^i}$, exactly equals its marginal cost,p_1^*. Condition 28b says that firm j's marignal benefit from selling an aditional unit of good 1, p_1^*, exactly equals its marginal cost, $\frac{d\,c^j(q^{j*})}{dq^j}$. Equations 28a determine each consumer's demand function. We can add them up to get aggregate demand, which is the LHS of the third equation. Equations 28b determine each firm's supply function. We can then add them to get aggregate supply, the RHS of the equation 28c.The third equation is thus the requirement that at the equilibrium price supply equals demand. The I+J+1 conditions in equation 28 characterize the I+J+1 equilibrium values and p_1^*. As long as $\max_i \frac{d\phi^i(0)}{dx_1^i} > \min_j \frac{dc^i(0)}{dq^j}$, the aggregate consumption and production of good 1 must be strictly positive in a competitive equilibrium. We will assume this is the case in the remainder of this section.

Notice that the equilibrium conditions involve neither the initial endowments of the consumers nor their ownership shares. Thus the equilibrium allocation of x_1 and the price of x are independent of the initial conditions. This follows directly from the assumption of quasilinear utility. However, since equilibrium allocations of the numeraire are found by using each consumer's budget constraint, the equilibrium allocations of the numeraire will depend on initial endowments and ownership shares.

2.4.2. *Constructive analysis of the model.*

1.: For each consumer, derive their Walrasian demand for the consumption good, $x_1^i(p)$. Add across consumers to derive the aggregate demand, $x_1(p) = \sum_{i=1}^{I} x_1^i(p)$. Because each demand curve is downward sloping, the aggregate demand curve will be downward sloping. Graphically, this addition is done by adding the demand curves "horizontally". Because demand curves are defined by the relation:

$$p = \frac{d\,\phi^i(x_1^i)}{dx_1^i} \tag{29}$$

the price at which each individual's demand curve intersects the vertical axis is $\frac{d\,\phi^i(x_1^i(0))}{dx_1^i}$, and gives that individual's marginal willingness to pay for the first unit of output. The intercept for the aggregate demand curve is therefore $\max_i \frac{d\phi^i(0)}{dx_1^i}$. Hence if different consumers have different $\phi^i(\cdot)$ functions, not all demand curves will have the same intercept, and the demand curve will become flatter as price decreases.

2.: For each firm, derive the supply curve for the consumption good, $q^j(p)$. Add across firms to derive the aggregate supply, $q(p)= \sum_{j=1}^{J} q^j(p)$. For each firm, the supply curve is given by

$$p = \frac{d\,c^j(q^j)}{dq^j} \tag{30}$$

Thus each firm's supply curve is the inverse of its marginal cost curve. Because we have assumed that $c^{j\prime\prime}(\cdot) < 0$, the supply curve will be upward sloping or flat. Again, addition is done by adding the supply curves horizontally. The intercept of the aggregate supply curve will be $\min_j \frac{dc^i(0)}{dq^j}$. If firms' cost functions are strictly convex, aggregate supply will be upward sloping.

3.: Find the price where supply equals demand: find p. such that $x(p^*) = q(p^*)$. Because the market clears for good l, it must also clear for the numeraire. The equilibrium point will be at the price and quantity where the supply and demand curves cross.

2.4.3. *A note on social cost and benefit.* The firm's supply function is $q^j(p)$ and satisfies $p = \frac{d\,c^j(q^j)}{dq^j}$. Thus at any particular price, firms choose their quantities so that the marginal cost of producing an additional unit of production is exactly equal to the price. Similarly, the consumer's demand function is $x_1^i(p)$ such that $p = \frac{d\,\phi^i(x_1^i)}{dx_1^i}$. Thus at any price, consumers choose quantities so that the marginal benefit of consuming an additional unit of x is exactly equal to its price. When both firms and consumers do this, we get that, at equilibrium, the marginal cost of producing an additional unit of x is exactly equal to the marginal utility of consuming an additional unit of x. This is true both individually and in the aggregate. Thus at the equilibrium price, all units where the marginal social cost is less than or equal to the marginal social benefit are produced and consumed, and no other units are. Thus the market acts to produce an efficient allocation.

2.5. **Graphical Analysis of the Partial Equilibrium Model.**

2.5.1. *The consumer.* Consider a consumer with utility function

$$v^i(x_1^i, m^i) \; = \; {x_1^i}^{\frac{1}{2}} \; + \; m^i \tag{31}$$

Assume that the price of x_1 is p_1 and that the price of the composite good m is one. Assume that the consumer has wealth w^i. This wealth is made up of an initial endowment of the composite good ω_m^i and profits from firms in which the consumer has ownership shares. The consumer problem is then

$$\begin{aligned} &\max_{x_1^i,\, m^i} \; {x_1^i}^{\frac{1}{2}} \; + \; m^i \\ &\text{such that } \; m^i \; + \; p_1^* \, x_1^i \; \leq \; w^i \end{aligned} \tag{32}$$

The Lagrangian function is given by

$$\mathcal{L} \; = \; {x_1^i}^{\frac{1}{2}} \; + \; m^i \; - \; \lambda(m^i \; + \; p_1 x_1^i \; - \; w^i) \tag{33}$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1^i} \; = \frac{1}{2} {x_1^i}^{\frac{-1}{2}} \; - \; \lambda \, p_1 \; = \; 0 \tag{34a}$$

$$\frac{\partial \mathcal{L}}{\partial m^i} \; = 1 \; - \; \lambda \; = \; 0 \tag{34b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \; = \; - \left[m^i \; + \; p_1 x_1^i \right] \; + \; w^i \; = \; 0 \tag{34c}$$

Substituting λ from equation 34b into equation 34a we obtain

$$\begin{aligned} \frac{1}{2} {x_1^i}^{\frac{-1}{2}} \; &= p_1 \\ \Rightarrow \; {x_1^i}^{\frac{-1}{2}} \; &= \; 2p_1 \\ \Rightarrow \; x_1^i \; &= \frac{1}{4p_1^2} \end{aligned} \tag{35}$$

The demand for x_1^i depends only on p_1. The demand for m is obtained from equation 34c.

$$\begin{aligned} \left[m^i \; + \; p_1 x_1^i \right] \; &= w^i \\ \Rightarrow \; \left[m^i \; + \; p_1 \, \frac{1}{4p_1^2} \right] \; &= w^i \\ \Rightarrow \; m^i \; = \; w^i \; - \; \frac{1}{4p_1} \end{aligned} \tag{36}$$

The demand for m is just the initial wealth minus the amount spent on x_1. For a $p_1 = \frac{1}{2}$ and initial wealth of 3, the optimum consumption point is represented in figure 2. By varying the price of x_1 we can trace out a demand curve for this consumer as in figure 3

FIGURE 2. Optimal Consumption for Consumer with Quasilinear Utility

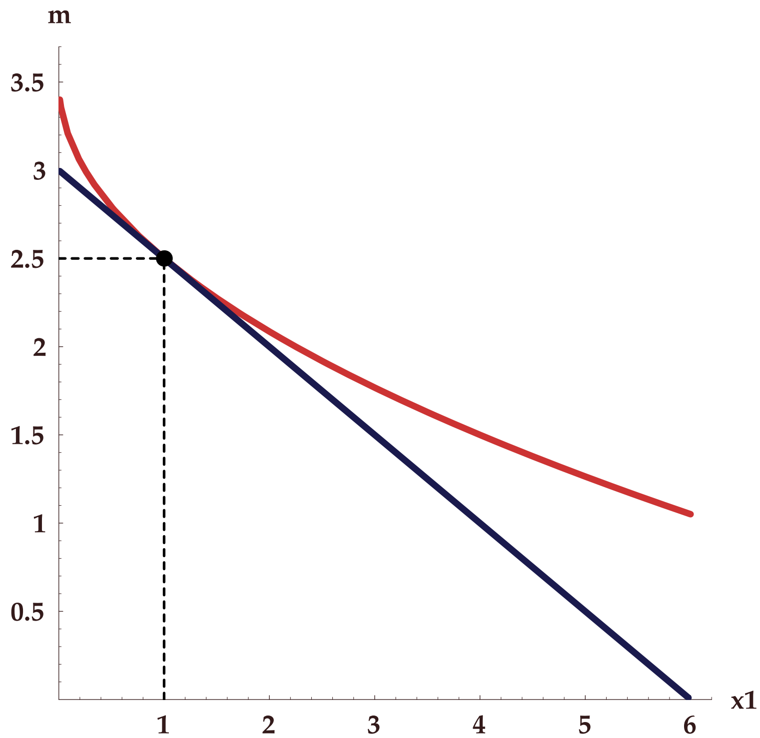
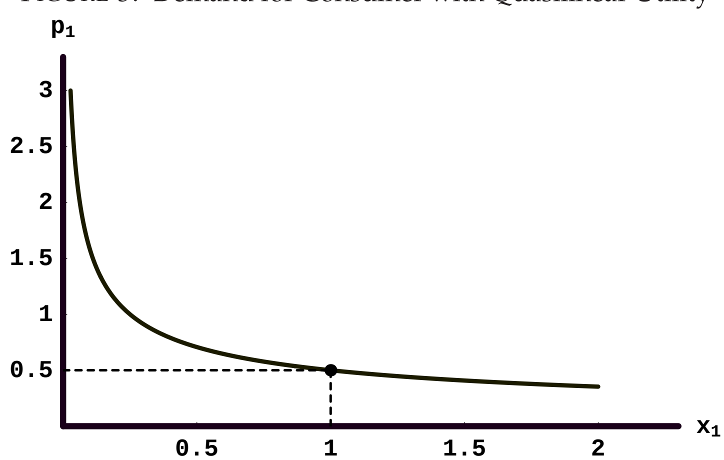


FIGURE 3. Demand for Consumer with Quasilinear Utility



Now consider figure 4 where we graph the demand functions for three consumers, each with a different utlity function. The price is p_1 and quantity demanded is x_1. The demand functions each have a different slope. In figure 5, we add the demand curves horizontally to obtain the aggregate demand function.

FIGURE 4. Demand Curves for 3 Consumers

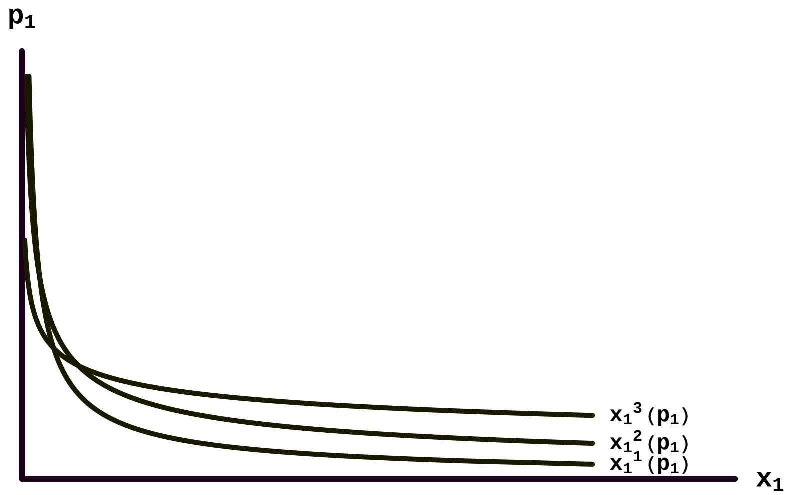
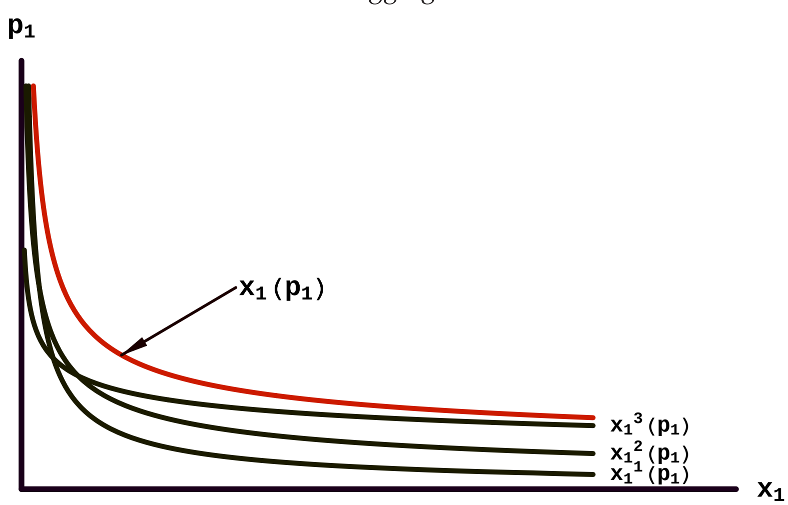


FIGURE 5. Aggregate Demand



2.5.2. *The firm.* Consider a firm with production function

$$q^j \;=\; z_1^{j\,\frac{1}{4}}\, z_2^{j\,\frac{1}{2}} \tag{37}$$

The firm uses two inputs z_1 and z_2 with fixed prices to produce the output q. We can think of z_1 and z_2 being among the goods represented by the composite good m. Assume that the price of z_1 is p_{z_1} and that the price z_2 is p_{z_2}. We will drop the j superscript for the time being for ease of notation. The Lagrangian for cost minimization is then

$$\mathcal{L} \;=\; p_{z_1} z_1 \;+\; p_{z_2}\, z_2 \;-\; \lambda\,(z_1^{\frac{1}{4}} z_2^{\frac{1}{2}} \;-\; \bar{q}\,) \tag{38}$$

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial z_1} = p_{z_1} - \frac{\frac{1}{4}\, q}{z_1}\, \lambda = 0 \tag{39a}$$

$$\frac{\partial \mathcal{L}}{\partial z_2} = p_{z_2} - \frac{\frac{1}{2}\, q}{z_2}\, \lambda = 0 \tag{39b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\, z_1^{\frac{1}{4}} z_2^{\frac{1}{2}} + \bar{q} = 0 \tag{39c}$$

Taking the ratio of the equations 39a and 39b we obtain

$$\frac{p_{z_1}}{p_{z_2}} = \frac{z_2}{2 z_1} \tag{40}$$

We can then solve for z_2 as

$$z_2 = \frac{2\, z_1\, p_{z_1}}{p_{z_2}} \tag{41}$$

Substituting in the equation 39c we obtain

$$\begin{aligned} \bar{q} &= z_1^{\frac{1}{4}} z_2^{\frac{1}{2}} \\ &= z_1^{\frac{1}{4}} \left(\frac{2\, z_1\, p_{z_1}}{p_{z_2}} \right)^{\frac{1}{2}} \\ &= 2^{\frac{1}{2}} \left(\frac{p_{z_1}}{p_{z_2}} \right)^{\frac{1}{2}} z_1^{\frac{3}{4}} \end{aligned} \tag{42}$$

Solving for z_1 we obtain

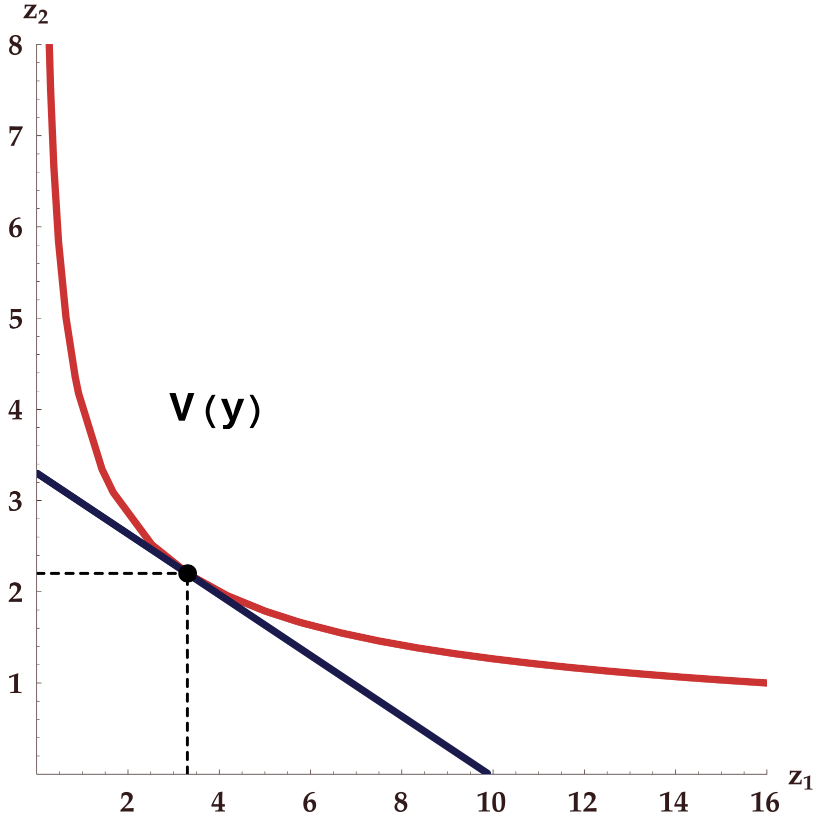
$$\begin{aligned} \bar{q} &= 2^{\frac{1}{2}} \left(\frac{p_{z_1}}{p_{z_2}} \right)^{\frac{1}{2}} z_1^{\frac{3}{4}} \\ \Rightarrow\ z_1^{\frac{3}{4}} &= \bar{q}\, 2^{\frac{-1}{2}} \left(\frac{p_{z_2}}{p_{z_1}} \right)^{\frac{1}{2}} \\ \Rightarrow\ z_1 &= \bar{q}^{\,\frac{4}{3}}\, 2^{\frac{-2}{3}} \left(\frac{p_{z_2}}{p_{z_1}} \right)^{\frac{2}{3}} \\ &= \bar{q}^{\frac{4}{3}} \left(\frac{1}{2} \right)^{\frac{2}{3}} \left(\frac{p_{z_2}}{p_{z_1}} \right)^{\frac{2}{3}} \end{aligned} \tag{43}$$

Similarly for z_2

$$z_2 = \bar{q}^{\frac{4}{3}}\, 2^{\frac{1}{3}} \left(\frac{p_{z_1}}{p_{z_2}} \right)^{\frac{1}{3}} \tag{44}$$

The optimal levels of input use when $p_{z_1} = 1$, $p_{z_2} = 3$ and $q = 2$ are pictured in figure 6
Now if we substitute for the z_1 and z_2 in the cost expression we obtain

FIGURE 6. Cost Minimization for the Firm



$$
\begin{aligned}
c &= p_{z_1}\, z_1 \;+\; p_{z_2}\, z_2 \\
&= p_{z_1}\left(\bar{q}^{\frac{4}{3}} \left(\frac{1}{2}\right)^{\frac{2}{3}} \left(\frac{p_{z_2}}{p_{z_1}}\right)^{\frac{2}{3}} \right) \;+\; p_{z_2}\left(\bar{q}^{\frac{4}{3}}\, 2^{\frac{1}{3}} \left(\frac{p_{z_1}}{p_{z_2}}\right)^{\frac{1}{3}} \right) \\
&= \bar{q}^{\frac{4}{3}} \left(\left(\frac{1}{2}\right)^{\frac{2}{3}} \left(p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}\right) \;+\; 2^{\frac{1}{3}} \left(p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}\right) \right) \\
&= \bar{q}^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \left(\left(\frac{1}{2}\right)^{\frac{2}{3}} \;+\; 2^{\frac{1}{3}} \right) \\
&= \bar{q}^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \left(2^{\frac{-2}{3}} \;+\; 2^{\frac{1}{3}} \right) \\
&= \bar{q}^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}\, 2^{\frac{-2}{3}} \left(2^{\frac{-2}{3}} 2^{\frac{2}{3}} \;+\; 2^{\frac{1}{3}} 2^{\frac{2}{3}} \right) \\
&= \bar{q}^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}\, 2^{\frac{-2}{3}}\, (1 \;+\; 2) \\
&= \frac{3}{2^{\frac{2}{3}}}\, \bar{q}^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}
\end{aligned} \tag{45}
$$

Marginal cost is given by

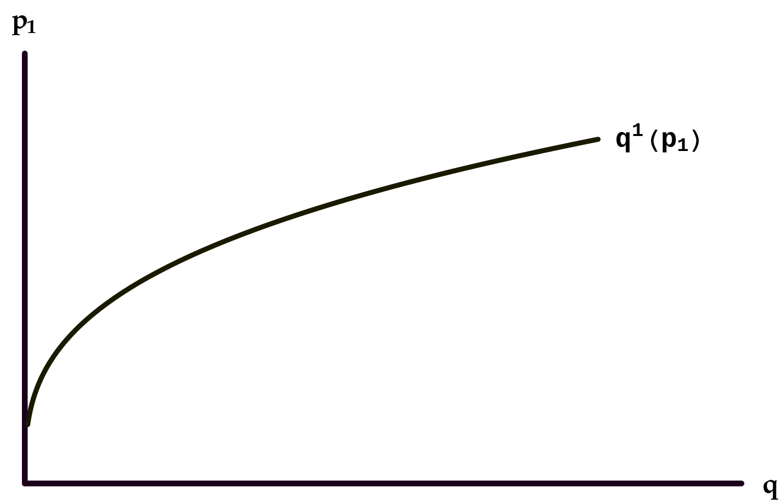
$$\begin{aligned} MC &= \frac{3\, 2^{\frac{-2}{3}}\, q^{\frac{4}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}}}{dq} = 4 \times 2^{\frac{-2}{3}}\, q^{\frac{1}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \\ &= 2^2 \times 2^{\frac{-2}{3}}\, q^{\frac{1}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \\ &= 2^{\frac{4}{3}}\, q^{\frac{1}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \end{aligned} \tag{46}$$

The firm maximizes profit by setting price equal to marginal cost. The price of q is p_1 if this is production of the commodity from section 2.5.1.

$$\begin{aligned} p_1 &= 2^{\frac{4}{3}}\, q^{\frac{1}{3}}\, p_{z_1}^{\frac{1}{3}} p_{z_2}^{\frac{2}{3}} \\ \Rightarrow q^{\frac{1}{3}} &= p_1\, 2^{\frac{-4}{3}}\, p_{z_1}^{\frac{-1}{3}}\, p_{z_2}^{\frac{-2}{3}} \\ \Rightarrow q &= p_1^3\, 2^{-4}\, p_{z_1}^{-1}\, p_{z_2}^{-2} \\ \Rightarrow q &= \frac{p_1^3}{16 p_{z_1} p_{z_2}^2} \end{aligned} \tag{47}$$

We can graph this supply function as in figure 7 for p_{z_1} =1 and p_{z_2} = 3. Also consider figure 8 where we graph the supply functions for three firms. The price is p_1 and quantity supplied is q. The supply functions are each different. In figure 9, we derive aggregate supply by adding the suuply curves horizontally to obtain the aggregate supply function.

FIGURE 7. Supply Curve for a Single Firm



2.5.3. *Market clearing.* Now consider a market where there are 5 consumers of each type and 10 firms of each type, or 15 consumers and 30 firms. Aggregate supply and demand are shown in figure 10. The price is p_1 and aggregate quantity is q. The equilibrium price occurs where the supply and demand curves are equal.

FIGURE 8. Supply Curves for 3 Firms

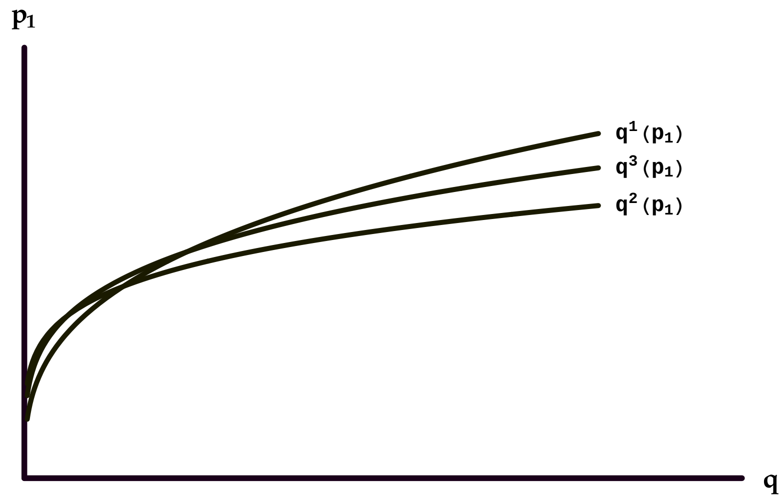
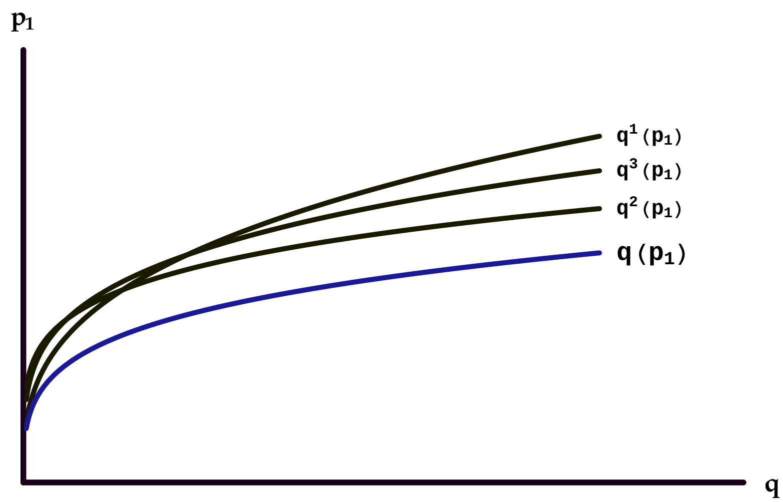


FIGURE 9. Aggregate Supply



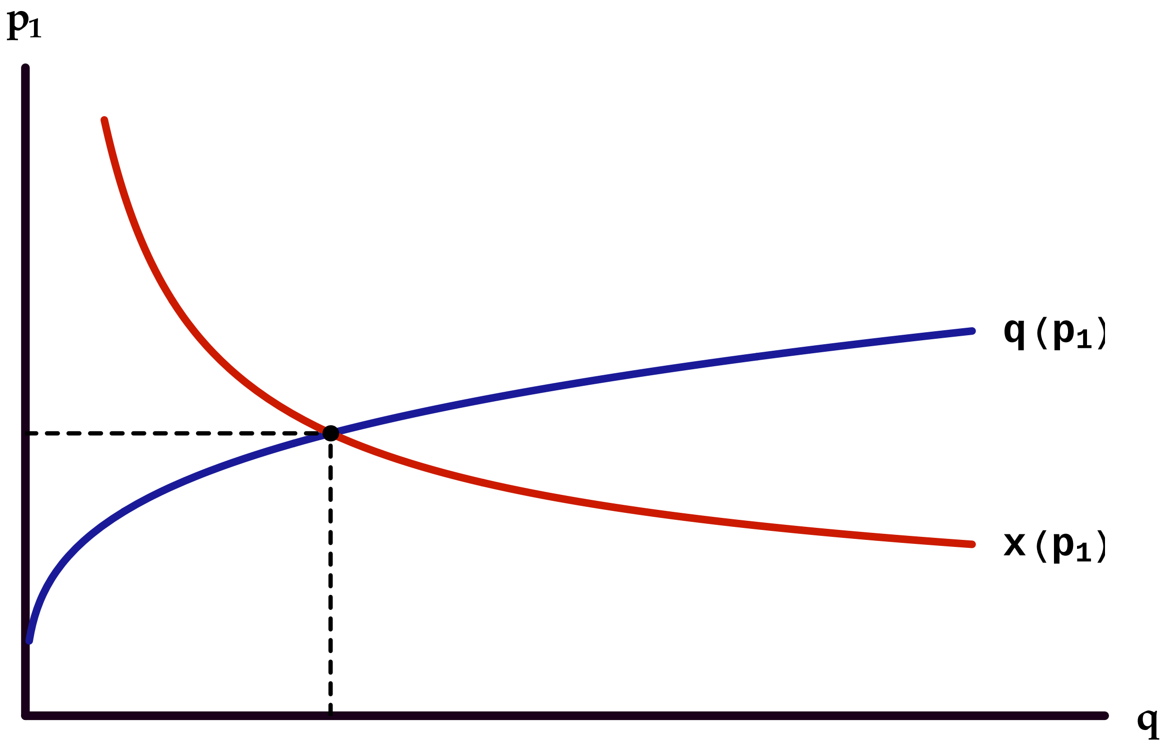
3. THE FUNDAMENTAL WELFARE THEOREMS OF ECONOMICS

We now turn to the efficiency properties of a market allocation. The basic questions are:

(i) When are the allocations made by markets ”efficient?
(ii) Is every efficient allocation the market allocation for some initial conditions?

These questions have to do with decentralization. When can we decentralize the decisions we make in our society? Will profit-maximizing firms and utility maximizing agents arrive at a Pareto optimal allocation through the market? If we have a particular Pareto optimal allocation is desired, can we rely on the market to obtain it, given we start at the right place (i.e., initial endowment for consumers)? The proper concept of efficiency is Pareto optimality. An allocation is Pareto optimal if there is no other feasible allocation that makes all agents at least as well off and some agent better off.

FIGURE 10. Aggregate Demand and Supply



3.1. **The First Fundamental Theorem of Welfare Economics in the Context of the Quasilinear Model.** The study of Pareto optimal allocations is greatly simplified in the context of the quasilinear partial equilibrium model. When preferences are quasilinear, the frontier of the utility possibility set is linear. That is, all points that are Pareto efficient involve the same consumption of the non-numeraire good by the consumers, and differ only in the distribution of the numeraire among the consumers. When one transfers one unit of the numeraire good from consumer i to consumer j, the utility of consumer i drops by one unit and utility of consumer j rises by one unit. To make this clear, suppose the consumption and production levels of good 1 are fixed at $(\bar{x}_1^1, \bar{x}_2^2, \ldots, \bar{x}_1^I, \bar{q}^1, \bar{q}^2, \ldots, \bar{q}^J)$. With these levels of production of q, there will be ω_m - $\sum_{j=1}^J c^j(\bar{q}^j)$ of the numeraire good m to be distributed among the consumers given that with p_m = 1, the number of units of m used by the j$^{\text{th}}$ firm is equivalent to the cost. The utility of consumer i is given by $\phi^i(\bar{x}_1^i) + m^i$. Because the numeraire can be traded one-for-one among consumers, the set of utilities that can be attained for the I consumers by distributing the available amounts of the numeraire (ω_m - $\sum_{j=1}^J c^j(\bar{q}^j)$ from equation 27) is given by

$$\left\{ (u^1, u^2, \ldots, u^I) : \sum_{i=1}^I u^i \;\leq\; \sum_{i=1}^I \phi^i(\bar{x}_1^i) \;+\; \omega_m \;-\; \sum_{j=1}^J c^j(\bar{q}^j) \right\} \tag{48}$$

This is the utility possibility set for any particular allocation of the good 1, $(\bar{x}_1^1, \bar{x}_2^2, \ldots, \bar{x}_1^I, \bar{q}^1, \bar{q}^2, \ldots, \bar{q}^J)$, if we allow the remaining numeraire to be distributed among consumers in any possible way. The boundary of this set is a hyperplane with normal vector (1,1,. . . ,1). For any initial endowment vector, the Pareto frontier will be those allocations that extend the boundary out as far as possible. The utility possibility set for the efficient allocation is the set generated by the those combinations of (u^1,u^2,..., u^I) that maximize the right hand side of equation 48. Specifically, the optimal consumption and production levels of good 1 can be obtained as the solution to

$$\begin{aligned} &\max_{\substack{(x_1^1, x_1^2, \ldots, x_1^I) \geq 0 \\ (q^1, q^2, \ldots, q^J) \geq 0}} \sum_{i=1}^{I} \phi^i(x_1^i) \;-\; \sum_{j=1}^{J} c^j(q^j) \;+\; \omega_m \\ &\text{such that} \sum_{i=1}^{I} x_1^i \;-\; \sum_{j=1}^{J} q^j \;=\; 0 \end{aligned} \tag{49}$$

We call the x's and q's generated by such a procedure the optimal production and consumption levels of good x. If the firms have strictly convex cost functions and $\phi^i(\cdots)$ is strictly concave, then there will be a unique (x, q) that maximizes the above expression. The value of the term $\sum_{i=1}^{I} \phi^i(x_1^i) \;-\; \sum_{j=1}^{J} c^j(q^j)$ is known as *Marshallian* aggregate surplus. One can think of it as the total utility generated from the consumption of good 1 minus the costs of producing it. The constraint is that total consumption of x_1 is the same as the total production q. We can set up a Langrangian problem to determine the optimal levels of (x_1^1,x_2^2, ...,x_1^I, q^1,q^2, ...,q^J).

$$\mathcal{L} \;=\; \sum_{i=1}^{I} \phi^i(x_1^i) \;-\; \sum_{j=1}^{J} c^j(q^j) \;+\; \omega_m \;-\; \mu \left(\sum_{i=1}^{I} x_1^i \;-\; \sum_{j=1}^{J} q^j \right) \tag{50}$$

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial x_1^i} \;=\; \frac{d\phi^i(x_1^i)}{dx_1^i} \;-\; \mu \;\leq\; 0, \quad i = 1, 2, \ldots, I \tag{51a}$$

$$\left(\frac{d\phi^i(x_1^i)}{dx_1^i} \;-\; \mu \right) x_1^i \;= 0, \quad i = 1, 2, \ldots, I \tag{51b}$$

$$\frac{\partial \mathcal{L}}{\partial q^j} \;=\; \frac{dc^j(q^j)}{dq^j} \;+\; \mu \;\geq\; 0, \quad j = 1, 2, \ldots, J \tag{51c}$$

$$\left(\frac{dc^j(q^j)}{dq^j} \;+\; \mu \right) q^j \;= 0, \quad j = 1, 2, \ldots, J \tag{51d}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} \;=\; -\sum_{i=1}^{I} x_1^i \;+\; \sum_{j=1}^{J} q^j \;=\; 0 \tag{51e}$$

Note that these are exactly the conditions as the conditions defining the competitive equilibrium in equations 28 except that p^* has been replaced by μ. We repeat those conditions for convenience here.

$$p_1^* \geq \frac{d\,\phi^i(x_1^{i*})}{dx_1^i}, \text{ with equality if } x_1^{i*} > 0,\ i = 1, 2, \ldots, I$$

$$p_1^* \leq \frac{d\,c^j(q^{j*})}{dq^j}, \text{ with equality if } q^{j*} > 0,\ j = 1, 2, \ldots, J$$

$$\sum_{i=1}^{I} x_1^{i*} = \sum_{j=1}^{J} q^{j*}$$

In other words, we know that the allocation produced by the competitive market satisfies these conditions, and that μ = p*. Thus the competitive market allocation is Pareto optimal, and the market clearing price p*is the shadow value of the constraint, i.e., the social benefit generated by consuming one more unit of output or producing one less unit of output. Again the result is that at p* the marginal social benefit of additional output equals the marginal social cost. We can state this result more formally as the first fundamental theorem of welfare economics for the partial equilibrium case.

Proposition 1 (First Fundamental Theorem of Welfare Economics)**.**

If the price p* *and allocation* $\left(x_1^{1*}, x_1^{2*}, \ldots, x_1^{I*}, q^{1*}, q^{2*}, \ldots, q^{J*}\right)$

constitute a competitive equilibrium, then this allocation is Pareto optimal.

The first theorem of welfare economics is a formal expression of Adam Smith's invisible hand the market acts to allocate commodities in a Pareto optimal manner. Since p* = μ, which is the shadow price of additional units of x, each firm acting in order to maximize its own profits chooses the output that equates the marginal cost of its production to the marginal social benefit, and each consumer, in choosing the quantity to consume in order to maximize utility, is also setting marginal benefit equal to the marginal social cost. The first welfare theorem holds quite generally whenever there are complete markets. It will fail, however, when there are commodities that have no markets as in the externalities or public goods.

3.2. **The Second Fundamental Theorem of Welfare Economics in the Context of the Quasilinear Model.** The second fundamental theorem of welfare economics is a converse to the first. That is, "Can any Pareto optimal allocation be generated as the outcome of a competitive market, for some suitable initial endowment vector?" The answer to this question is yes. To see why, recall that when all $\phi^i(\cdot)$'s are strictly concave and all c$^j(\cdot)$'s are strictly convex, there is a unique allocation of the consumption commodity x_1 that maximizes the sum of the consumers' utilities as in equation 50. The set of Pareto optimal allocations is derived by allocating the consumption commodity in this manner and then varying the amount of the numeraire commodity given to each of the consumers. The set of Pareto optimal allocations is along a line with normal vector (1,1,1 . . . ,1) because one unit of utility can be transferred from one consumer to another by transferring a unit of the numeraire. Thus any Pareto optimal allocation can be generated by letting the market work and then appropriately transferring the numeraire to reach the Pareto optimal point. But, recall that in the quasilinear model, firms' production decisions and consumers' consumption decisions do not depend on the initial endowment of the numeraire. Because of this, one could also perform the transfers before the market works. This allows implementation of any point along the Pareto frontier by suitable arrangement of the initial endowments of ω_m^i.

To see why, let (x_1^*, q*) be the Pareto optimal allocation of the consumption commodity. Now suppose we want each consumer to get ($x_1^{i\,*}$,$m^{i\,*}$) after the market equilibrium takes place. Note that $\sum_{i=1}^{I} m^{i\,*}$ =

ω_m - $\sum_{j=1}^{J} c^j(\bar{q}^j)$ If consumer i is to have m^{i^*} units of the numeraire after the transfer, he needs to have $\bar{m}^i$ before the transfer, where $\bar{m}^i$ is implicitly defined by

$$\bar{m}^i \; + \; \sum_{j=1}^{J} \theta_j^i \left(p_1^* \, q^{j*} \; - \; c^j(q^{j*}) \right) \; = \; m^{i^*} \; + \; p_1^* x_1^{i^*} \tag{52}$$

Hence if each individual has wealth $\bar{m}^i$ before the market starts to work, the allocation (x_1^*, q^*, m^*) will result. This yields the second fundamental theorem of welfare economics.

Proposition 2 (Second Fundamental Theorem of Welfare Economics)**.**

For any Pareto optimal levels of utility $(u^{1^*}, u^{2^*}, \ldots, u^{I^*})$, there are transfers of the numeraire commodity $\left(T^1, T^2, \ldots, T^I\right)$) satisfying $\sum_{i=1}^{I} T^i = 0$, such that a competitive equilibrium reached from the endowments $\left(\omega_m^1 + T^1, \omega_m^2 + T^2, \ldots, \omega_m^I + T^I\right)$ yields precisely the utilities $(u^{1^*}, u^{2^*}, \ldots, u^{I^*})$

4. Welfare Analysis in the Partial Equilibrium Model

4.1. **Welfare and aggregate demand.** Recall that in a quasilinear model,Hicksian and Walrasian demand curves are the same so that the area to the left of the Walrasian demand curve is the same as the area to the left the Hicksian demand curve between the initial and final prices. When utility has the Gorman polar form (of which qausilinearity is a special case) aggregate demand can be written as a function of aggregate income or wealth. In this case aggregate demand can be treated as if it arose from the utility function of a representative consumer who faces prices p and has income m = $\sum_{i=1}^{I} \; m_i$ with indirect utility function

$$\begin{aligned} \psi(p, \, m) \; &= \frac{m \; - \; \sum_{i=1}^{I} \; f_i(p)}{g(p)} \\ &= \frac{1}{g(p)} \, m \; - \; \frac{\sum_{i=1}^{I} f_i(p)}{g(p)} \end{aligned} \tag{53}$$

The analogous indirect utility function for quasilinear preferences in the two good model is of the form

$$\begin{aligned} \psi(p_1, m) \; &= m \; - \; \theta(p_1) \\ &= m \; + \; \phi(p_1) \end{aligned} \tag{54}$$

Hence the area to the left of the Walrasian demand curve is a good measure of changes in social welfare.

4.2. **Marshallian Aggregate Social Surplus.** Consider the welfare or utility available from a given allocation $(x_1^1, x_2^2, \ldots, x_1^I, q^1, q^2, \ldots, q^J)$of good 1 across consumers and firms from equation 48 which we repeat here

$$\left\{ (u^1, u^2, \ldots, u^I) : \sum_{I=1}^{I} u^i \; \leq \; \sum_{i=1}^{I} \phi^i(x_1^i) \; + \; \omega_m \; - \; \sum_{j=1}^{J} c^j(q^j) \right\} \tag{48'}$$

Now consider a central planner who is redistributing the numeraire to maximize social welfare W(u^1, u^2, ..., u^I). The social welfare function W(u^1, u^2, ..., u^I)gives a level of welfare associated with any utility vector and allows him or her to compare any two distributions of utility in terms of their overall social welfare. Clearly the optimized level of W will be higher when the utility possibility set is larger. The social planner can therefore maximize social welfare by maximizing the right hand side of equation 48. The problem is then as follows

$$\begin{aligned} &\max_{\substack{(x_1^1, x_1^2, \ldots, x_1^I) \geq 0 \\ (q^1, q^2, \ldots, q^J) \geq 0}} \sum_{i=1}^{I} \phi^i(x_1^i) \; - \; \sum_{j=1}^{J} c^j(q^j) \; + \; \omega_m \\ &\text{such that} \sum_{i=1}^{I} x_1^i \; - \; \sum_{j=1}^{J} q^j \; = \; 0 \end{aligned} \tag{55}$$

The last term in the objective function is the initial aggregate endowment of the numeraire good, which is just a constant. The first two terms represent the difference between aggregate utility from consumption and aggregate cost of production. This difference is the societal benefit from good 1 since all profits return to consumers). Hence a change in the production and consumption levels of good 1 leads to an increase in welfare (with appropiate redistribution of the numeraire) if and only if it increases the Marshallian aggregate surplus.

$$S(x^1, x^2, \ldots, x^I, q^1, q^2, \ldots, q^J) \; = \; \sum_{i=1}^{I} \phi^i(x_1^i) \; - \; \sum_{j=1}^{J} c^j(q^j) \tag{56}$$

In many circumstances, the Marshallian surplus has a convenient and intuitive formulation in terms of the area lying vertically between the aggregate supply and demand functions for good 1. In particular, consider an aggregate demand $x_1(p) = \sum_{i=1}^{I} x_1^i(p)$ that is distributed optimally across the I consumers in the economy. Denote inverse aggregate demand by $P(x_1)$, that is for a specific level of aggregate demand $\bar{x}_1$ we have $P(\bar{x}_1) = x^{-1}(\bar{x}_1)$. Specifically, when each consumer optimally chooses his demand for good 1 at this price, total demand exactly equals $\bar{x}_1$. And at these individual demand levels (assuming we have an interior solution) each consumer's marginal benfit from consuming good 1 is exactly equal to its price, i.e., $P(x_1) = \frac{d\,\phi^i(x_1^i)}{dx_1^i}$. This is the aggregate level analogue of equation 29 for the individual consumer. This will be satisified when all consumers act as price takers and all consumers face the same price.

Similarly, consider an aggregate supply $q = \sum_{j=1}^{J} q^j$ that is distributed optimally across the J firms in the economy. This will be achieved when the marginal cost of production is the same for all firms at the individual levels of production that sum to industry supply and when these individual levels of marginal cost equal to industry level marginal cost. Specifically this occurs when $\frac{d\,c^j(q^j)}{dq^j} = \frac{d\,C(q)}{dq}$ where $\frac{d\,C(q)}{dq}$ is the industry marginal cost function which is the same as the industry inverse supply function $q^{-1}(\cdot) = C'(\cdot)$.

To see why individual firm marginal costs are all equal consider merging all of the firms in the industry into one large multiplant firm. Denote the profit function of the i^{th} independent firm by

$$\begin{aligned} \pi^j(p) \; &= \; \max_{y^j} \left[\sum_{\ell=1}^{L} p_\ell \, y_\ell^j \right] \; \textit{such that} \; [y^j \; \in \; Y^j \,] \\ y^j(p) \; &= \text{argmax}_{y^j} \left[\sum_{\ell=1}^{L} p_\ell \, y_\ell^j \right] \; \textit{such that} \; [y^j \; \in \; Y^j \,] \end{aligned} \tag{57}$$

Now assume that these are all single output firms each producing the same output. When the firms behave competitively and each maximizes profit, aggregate supply of this product is given by

$$y(p) \;=\; \sum_{j=1}^{J} y^j(p) \tag{58}$$

Now consider the aggregate production possibility set

$$Y \;=\; Y^1 + Y^2 + \cdots + Y^J \;=\; \{y \in R^L : y = \sum_{j=1}^{j} y^j \text{ for some } y^j \in Y^j, j = 1, 2, \ldots, J\} \tag{59}$$

The aggregate peoduction set Y describes the production vecors that are feasible in the aggregate if all the production sets are used together. Let π(p) and y(p) be the profit function and supply function of the aggregate production set. These are the profit function and supply function that would arise if a single price taking firm were to operate all the indidivual production sets. Now consider the following proposition.

Proposition 3. *For all p >> 0, we have*

(i) $\pi(p) = \sum_{j=1}^{J} \pi^j(p)$

(ii) $y(p) = \sum_{j=1}^{J} y^j(p)$ $(= \{\sum_{j=1}^{J} y \; : y^j \in y^j(p)$ *for every j*$\})$

Proof.

(i) For the first equality, note that if we take any collection of production plans y^j $\in$ Y^j, j = 1, 2, $\ldots$, J then $\sum_{j=1}^{j} y^j inY$. Because $\pi(\cdot)$ is the profit function for associated with Y, we have $\pi(\cdot) \geq p \sum_{j=1}^{J} y^j = \sum_{j=1}^{J} py^j$. It therefore follows that $\pi(p) \geq \sum_{j=1}^{J} \pi^j(p)$.

In the other direction, consider any y $\in$ Y. By the definition of the set Y, there are y^j $\in$ Y^j, j = 1,2, $\ldots$, J such that $\sum_{j=1}^{J} y^j = y$. So py = $p \sum_{j=1}^{J} y^j = \sum_{j=1}^{J} py^j \leq \sum_{j=1}^{J} \pi^j(p)$ for all y $\in$ Y. Thus $\pi(p) \leq sum_{j=1}^{J} \pi^J(p)$. These two inequalities, together, imply that $\pi(p) = \sum_{j=1}^{J} \pi^j(p)$.

(ii) For the second equality we must show that $\sum_{j=1}^{J} y^j(p) \subset y(p)$ and that $y(p) \subset \sum_{j=1}^{J} y^j(p)$. For the first relation, consider any set of individual production plans $y^j \in y^j(p)$, j = 1,2, $\ldots$, J. Then $p \sum_{j=1}^{J} y^j = \sum_{j=1}^{J} py^j = \sum_{j=1}^{J} \pi(p) = \pi(p)$. The last equality follows from the first part of the proposition. Hence, $\sum_{j=1}^{J} y^j \in y(p)$ and therefore $\sum_{j=1}^{J} y^j(p) \subset y(p)$.

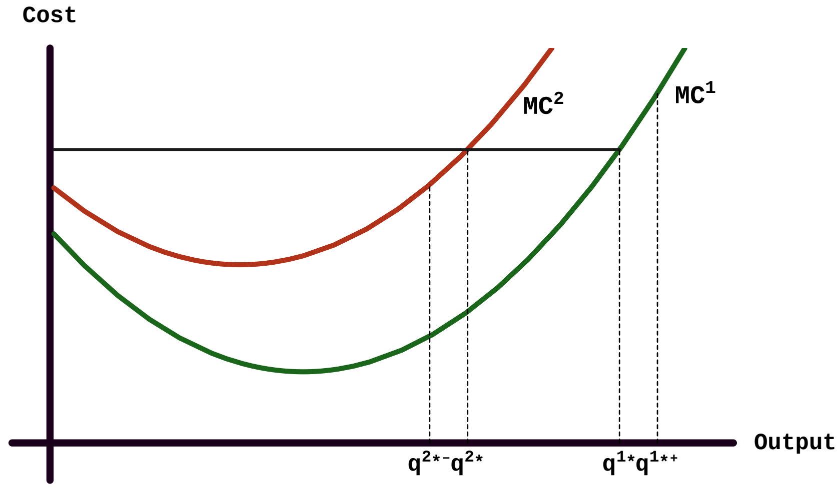
In the other direction take any y $\in$ y(p). Then y = $\sum_{j=1}^{J} y^j$ for some y^j $\in$ Y^j, j=1,2,. . . ,J. Because $p \sum_{j=1}^{J} y^j = \pi(p) = \sum_{j=1}^{J} \pi^j(p)$ and, for every j, we have $py^j \leq \pi^j(p)$ it must be that $py^j = \pi^j(p)$ for every j. Thus, $y^j \in y^j(p)$ for all j, and so $y \in \sum_{j=1}^{J} y^j(p)$. Thus we have $y(p) \subset \sum_{j=1}^{J} y^j(p)$.

□

The results tells us that if firms are mximizing profit facing output price p and factor prices w, then their supply behavior maximizes aggregate profit. But this means that if q = $\sum_{j=1}^{j} q^j$ is the aggregate output produced by the firms (q represents the positive elements of y), then the total cost of production is exactly equal to c(q,w), the value of the aggregate cost function which is obtained by minimzing the cost of production for the vector q given the aggregate production set Y. Thus the allocation of the production level q among the firms is cost minimizing. In the aggregate problem profit maximization implies that p = $\frac{dc(q,w)}{dq}$ for a single output firm. Similarly for an individual firm p = $\frac{dc^j(q,w)}{dq^j}$. But given the proposition this implies $\frac{dc^j(q,w)}{dq^j}$ = $\frac{dc(q,w)}{dq}$.

We can see this more clearly in figure 11. At the equilibrium, the first firm will produce q^{1*} and the second will produce q^{2*}. If we move some production from the second firm to the first firm we will be at the points q^{1*+} and q^{2*-} where q^1 is higher and q^2 is lower. This leads to an increase in cost of the area below MC^1 and between q^{1*} and q^{1*+} because the area under a marginal cost curve is a measure of total variable cost. This increase in cost is larger than the decrease in cost which is given by the area below MC^2 and between q^{2*-} and q^{2*}.

FIGURE 11. Minimizing Total Production Cost



It is not required that the price faced by consumers and firms be the same for Marshallian surplus to represent aggregate welfare. Consider figure 12. Note that we can break the surplus down into four parts:

1. a. Some of the surplus comes from consumption, $\sum_{i=1}^{I} \phi^i(x_1^i)$
   b. Some surplus is lost due to paying price p_1 for the good.

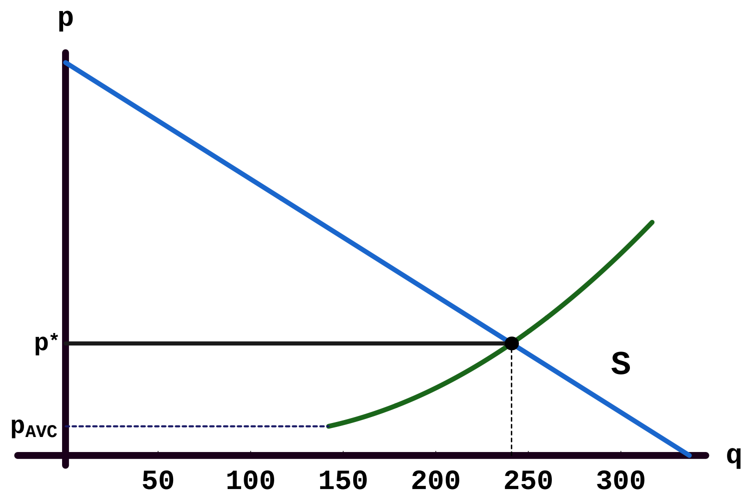
   Aggregate consumer surplus is then $\sum_{i=1}^{I} \left(\phi^i(x_1^i) \; - \; p_1 x_1^i\right)$. This corresponds to the utilitarian social welfare problem.
2. a. Some of the surplus is gained by firms in the form of revenue, $\sum_{j=1}^{J} p_1 q^j$.
   b. Some of the surplus is lost by the firms in the from of production cost, $\sum_{j=1}^{J} c^j(q^j)$.

   $\sum_{j=1}^{J} p_1 q^j$ - $\sum_{j=1}^{J} c^j(q^j)$ is the aggregate producer surplus, which is then redistributed to consumers in the form of dividends, $\theta_i^j \left(p_1 q^j - c^j(q^j)\right)$. So, consumers receive part of the benefit through consumption of the non-numeraire good, $\sum_{i=1}^{I} \left(\phi^i(x_1^i) \; - \; p_1 x_1^i\right)$ and part of the benefit through consumption of the dividends, which are measured in units of the numeraire: $\theta_i^j \left(p_1 q^j - c^j(q^j)\right)$.

Aggregate surplus is found by adding these two together, and noting that $\sum_{j=1}^{J} p_1 q^j$ = $\sum_{i=1}^{I} p_1 x_1^i$.

Now, how does Marshallian aggregate surplus change when the quantity produced and consumed changes? Let S(x, q) be the Marshallain aggregate surplus formally defined as follows:

FIGURE 12. Aggregate Surplus



$$S(x^1, x^2, \ldots, x^I, q^1, q^2, \ldots, q^J) \;=\; \sum_{i=1}^{I} \; \phi^i(x_1^i) \;-\; \sum_{j=1}^{J} c^j(q^j) \tag{60}$$

Consider a differential increase in consumption and production: (dx_1^1, dx_1^2, ...,dx_1^I, dq^1, ..., dq^J) satisfying $\sum_{j=1}^{J} dq^j \;=\; \sum_{i=1}^{I} x_1^i$. Note that under such a change, we increase total production and total consumption by the same amount. The differential in S is given by

$$dS \;=\; \left(\sum_{i=1}^{I} \frac{d\phi^i(x_1^i)}{dx_1^i}\, dx_1^i \right) \;-\; \left(\sum_{j=1}^{J} \frac{dc^j(q^j)}{dq^j}\, dq^j \right) \tag{61}$$

Because consumers maximize utility, $\frac{d\phi^i(x_1^i)}{dx_1^i} \;=\; p(x)$ for all i, and because producers maximize profit, $\frac{dc^j(q^j)}{dq^j} = c'(q)$ for all j. Thus

$$dS \;=\; \left(p(x) \sum_{i=1}^{I} \; dx_1^i \right) \;-\; \left(c'(q) \sum_{j=1}^{J} dq^j \right) \tag{62}$$

which by definition of our changes (and market clearing) implies

$$dS \;=\; (p(x) \;-\; c'(q))\, dx \tag{63}$$

And, integrating this from 0 to $\bar{x}$ yields

$$S \;=\; \int_0^{\bar{x}} (p(s) \;-\; c'(s))\, ds \tag{64}$$

Thus the total surplus is the area between the supply and demand curves between 0 and the quantity sold,$\bar{x}$.

4.3. **Example Calculations for Cost, Variable Cost, and Marginal Cost.** Marginal cost is given by the derivative of the variable cost function or

$$MC(q,\, w_1,\, w_2,\, \cdots,\, w_n) \;=\; \frac{\partial VC(q,\, w_1,\, w_2,\, \cdots w_n,\,)}{\partial\, q} \;=\; \frac{d\, VC(q)}{dq} \tag{65}$$

The last expression holds if we assume that all input prices and fixed inputs (z) do not change. The fundamental theorem of calculus says that for functions MC(q) that are continuous on a interval [a, b] the function VC defined on [a, b] by

$$VC(q) \;=\; \int_a^y \, MC(t)dt \tag{66}$$

is continuous on [a, b], differentiable on (a, b), and has derivative

$$\frac{d\,VC(q)}{dq} \;=\; MC(q), \quad \forall\, y \,\in\, (a, b) \tag{67}$$

What this says is that we can obtain the variable cost function by integrating the marginal cost function. For example if marginal cost is given by

$$MC(q) \;=\; 6 \;-\; 0.8q \;+\; .06q^2 \tag{68}$$

then we can obtain variable cost by integration

$$\begin{aligned} VC(q) \;&=\; \int_0^q \, MC(t)\, dt \\ &=\; \int_0^q \, (6 \;-\; 0.8t \;+\; .06t^2)\, dt \\ &= (6t \;-\; 0.4t^2 \;+\; 0.02t^3) \mid_0^q \\ &= (6q \;-\; 0.4q^2 \;+\; 0.02q^3) \;-\; 0 \\ &= 6q \;-\; 0.4q^2 \;+\; 0.02q^3 \end{aligned} \tag{69}$$

Then for q = 9, 10 and 11 we obtain as measures of variable cost

$$\begin{aligned} VC(9) \;&= 6(9) \;-\; 0.4(9)^2 \;+\; 0.02(9)^3 \\ &= 54 \;-\; 32.4 \;+\; .14.58 \;=\; 36.18 \\ VC(10) \;&= 6(10) \;-\; 0.4(10)^2 \;+\; 0.02(10)^3 \\ &= 60 \;-\; 40 \;+\; 20 \;=\; 40 \\ VC(11) \;&= 6(11) \;-\; 0.4(11)^2 \;+\; 0.02(11)^3 \\ &= 66 \;-\; 48.4 \;+\; 26.62 \;=\; 44.22 \end{aligned} \tag{70}$$

If the price is \$4, a firm will set price equal to marginal cost and produce 10 units. This is obvious as follows

$$\begin{aligned} MC(q) \;=\; 6 \;-\; 0.8q \;+\; .06q^2 \;&= 4 \\ \Rightarrow\; 0.06q^2 \;-\; 0.8q \;+\; 2 \;&= 0 \end{aligned} \tag{71}$$

Using the quadratic formula we obtain

$$\begin{aligned}
0.06q^2 \ - \ 0.8q \ + \ 2 \ &= 0 \\
\Rightarrow \ q \ = \ &- \frac{(-0.8) \ \pm \ \sqrt{(-0.8)^2 \ - \ 4(0.06)(2)}}{2(0.06)} \\
&= - \frac{(-0.8) \ \pm \ \sqrt{0.64 \ - \ 0.48}}{0.12} \\
&= - \frac{(-0.8) \ \pm \ \sqrt{0.16}}{0.12} \\
&= \frac{-(-0.8) \ \pm \ 0.4}{0.12} \\
&= \frac{1.2}{.12}, \ \frac{0.4}{.12} \\
&= 10 \quad \text{or} \quad 3.333
\end{aligned} \tag{72}$$

Thus a firm will produce 10 units with a price of 4. Two identical firms will each produce 10 units for a total of 20 units and a total cost of \$80. If instead one firm produced 9 and the other 11 for a total of 20 units, the total cost would be \$80.40.

4.4. **Producer Surplus.** Producer surplus is the largest amount that could be subtracted from a firm's revenues and yet leave the firm willing to supply the product. Alternatively it is the difference between the amount the seller receives for selling a single unit and the cost of producing it. Total producer surplus in a market is then measured by summing up this difference over each unit of the good sold. Given an upward-sloping marginal cost (and supply) curve this is the area below the competitive market price and above the supply curve. Graphically it is the area Ope in the diagram below.

There are three alternative ways to obtain producer surplus for a competitive firm. If we have complete data on the cost function and the output price then producer surplus is given by

$$\begin{aligned}
Producer\ Surplus \ &= Revenue \ - \ Variable\, Cost \\
&= pq \ - \ C(q, w) \ + \ Fixed\, Cost
\end{aligned} \tag{73}$$

An alternative way is to compute revenue as in the previous expression and then compute variable cost by integrating marginal cost from 0 to the profit maximizing level of output at the given price. This yields

$$\begin{aligned}
Producer\ Surplus \ &= Revenue \ - \ Variable\, Cost \\
&= pq \ - \ \int_0^q \ MC(q, w)\, dq
\end{aligned} \tag{74}$$

A third way is to set marginal cost equal to price and solve the equation for output q as a function of price p and then integrate this expression from the price down to where the function intersects the vertical axis. This gives

$$\begin{aligned}
Producer\ Surplus \ &= Revenue \ - \ Variable\, Cost \\
&= \int_{a_0}^{p} \ q(p, w) dp
\end{aligned} \tag{75}$$

where a_0 is the intercept of marginal cost or supply equation with the vertical axis.

As an example consider the cost function

$$cost(q, w_1, w_2, \cdots, w_n) = 400 + 16y + y^2 \tag{76}$$

with a market price of \$80. Marginal cost is given by

$$MC(q, w_1, w_2, \cdots, w_n) = 16 + 2q \tag{77}$$

Setting price equal to marginal cost we obtain

$$\begin{aligned} p &= 16 + 2q = MC(q, w_1, w_2, \cdots, w_n) \\ \Rightarrow p &= 16 + 2q \\ \Rightarrow 2q &= p - 16 \\ \Rightarrow q(p) &= \frac{1}{2}p - 8 \end{aligned} \tag{78}$$

We can find the intercept on the price axis by setting q equal to zero and solving the equation for p. Doing so we obtain

$$\begin{aligned} q(p) &= \frac{1}{2}p - 8 \\ \Rightarrow \frac{1}{2}p &= q(p) + 8 \\ \Rightarrow p &= 2q(p) + 16 \\ a_0 &= (2)(0) + 16 \\ &= 16 \end{aligned} \tag{79}$$

With a price of 80, the firm will produce 32 units of output.

$$\begin{aligned} q(p) &= \frac{1}{2}p - 8 \\ &= \frac{1}{2}(80) - 8 \\ &= 40 - 8 = 32 \end{aligned} \tag{80}$$

Variable cost for the firm with 32 units of output is

$$\begin{aligned} VC(q, w_1, w_2, \cdots, w_n) &= 16q + q^2 \\ &= (16)(32) + 32^2 \\ &= 512 + 1024 \\ &= 1536 \end{aligned} \tag{81}$$

Revenue is price times quantity which gives

$$\begin{aligned} R(p, q) &= pq \\ &= (80)(32) \\ &= 2560 \end{aligned} \tag{82}$$

This gives producer surplus of

$$
\begin{aligned}
Producer\ Surplus &= Revenue\ -\ Variable\ Cost \\
&= 2560\ -\ 1536 \\
&= 1024
\end{aligned}
\tag{83}
$$

Given the marginal cost equation we can obtain variable cost by integration as follows

$$
\begin{aligned}
VC(q)\ &=\ \int_0^q\ MC(t)\,dt \\
&=\ \int_0^q\ 16\ +\ 2\,t\,dt \\
&= (16t\ +\ t^2)\ |_0^q \\
&= (16q\ +\ q^2)\ -\ 0 \\
&= 16q\ +\ q^2
\end{aligned}
\tag{84}
$$

Producer surplus is as before. We can also integrate the supply function from 16 to 80 to obtain producer surplus. This will give

$$
\begin{aligned}
PS(p)\ &=\ \int_{a_0}^p\ q(t)dt \\
&=\ \int_{16}^{80}\ q(p)\,dp \\
&=\ \int_{16}^{80}\ \frac{1}{2}p\ -\ 8\,dp \\
&= \frac{1}{4}\,p^2\ -\ 8\,p\ |_{16}^{80} \\
&= \left[\frac{1}{4}\,80^2\ -\ (8)\,(80)\right]\ -\ \left[\frac{1}{4}\,16^2\ -\ (8)\,(16)\right] \\
&= \left[\frac{1}{4}\,(6400)\ -\ 640\right]\ -\ \left[\frac{1}{4}\,(256)\ -\ 128\right] \\
&= [\,1600\ -\ 640\,]\ -\ [\,64\ -\ 128\,] \\
&= [\,960\,]\ -\ [\,-64\,] \\
&= 1024
\end{aligned}
\tag{85}
$$

Given that this example is linear, we also obtain this graphically using the area of a triangle where the base is 64 (80 - 16) and the height is 32. The area is then $\frac{1}{2}$ (base)(height) = $\frac{1}{2}$ (64) (32) = 1024.

4.5. **Examples of Partial Equilibrium Welfare Analysis.**

4.5.1. *Consumer.* Consider a consumer with utility function

$$
v^i(x^i, m^i)\ =\ m^i\ +\ 200x^i\ -\ 25{x^i}^2
\tag{86}
$$

Assume that the price of x is p and that the price of the composite good m is one. Assume that the consumer has wealth w^i. This wealth is made up of an initial endowment of the composite good ω_m^i and profits from firms in which the consumer has ownership shares. Drop the superscripts refering to the ith consumer for the time being. The consumer problem is then

$$\begin{aligned} &\max_{x,\, m}\ m\ +\ 200x\ -\ 25x^2 \\ &\text{such that}\ m\ +\ px\ \leq\ w \end{aligned} \tag{87}$$

The Lagrangian function is given by

$$\mathcal{L}\ =\ m\ +\ 200x\ -\ 25x^2\ -\ \lambda(m\ +\ px\ -\ w) \tag{88}$$

The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial x} = 200\ -\ 50x\ -\ \lambda\, p =\ 0 \tag{89a}$$

$$\frac{\partial \mathcal{L}}{\partial m} = 1\ -\ \lambda\ =\ 0 \tag{89b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda}\ =\ -\,[m\ +\ px]\ +\ w\ =\ 0 \tag{89c}$$

Substituting λ from equation 89b into equation 89a we obtain

$$\begin{aligned} 200\ -\ 50x\ &= p \\ \Rightarrow\ 50x\ &=\ 200\ -\ p \\ \Rightarrow\ x\ &= 4\ -\ .02p \end{aligned} \tag{90}$$

The demand for x depends only on p. The demand for m is obtained from equation 89c.

$$\begin{aligned} [m\ +\ px]\ &= w \\ \Rightarrow\ \left[m\ +\ 4p\ -\ .02p^2\right]\ &= w \\ \Rightarrow\ m\ =\ w\ -\ 4p\ +\ .02p^2 \end{aligned} \tag{91}$$

The demand for m is just the initial wealth minus the amount spent on x. By varying the price of x we can trace out a demand curve for this consumer as in figure 13.

Now consider an economy with 100 consumers of the type in equation 86. Aggregate demand is given by

$$Q\ =\ 100x\ =\ 400\ -\ 2p \tag{92}$$

This is shown in figure 14.

Consumer surplus is defined as the maximum amount which a consumer would be willing to spend, above the actual price, to consume the units he purchases. Total consumer surplus in a market is then measured by summing this difference over each unit of the good bought in the market. Graphically it is the area below the demand curve and above the price paid or (100,200,e) in figure 15.. One can obtain it using the area of a triangle in the case of linear demand. In general one can integrate the demand function from the price to the intercept of the inverse demand curve with the price axis. Or one can find the total

FIGURE 13. Demand for x by One Consumer

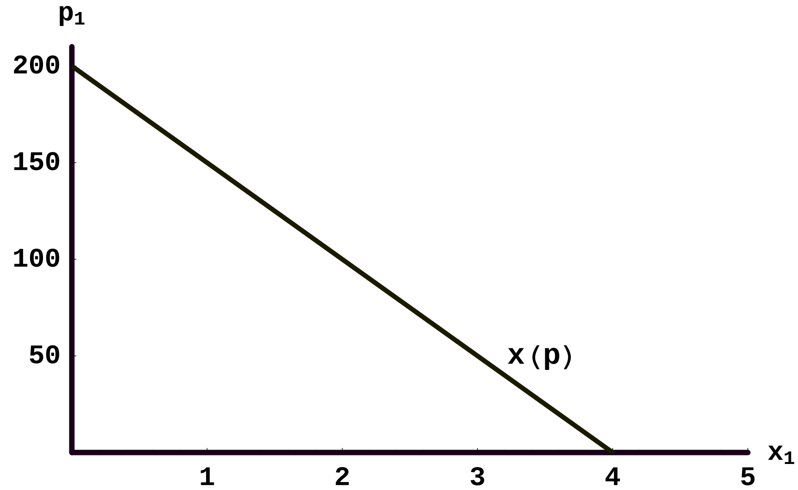
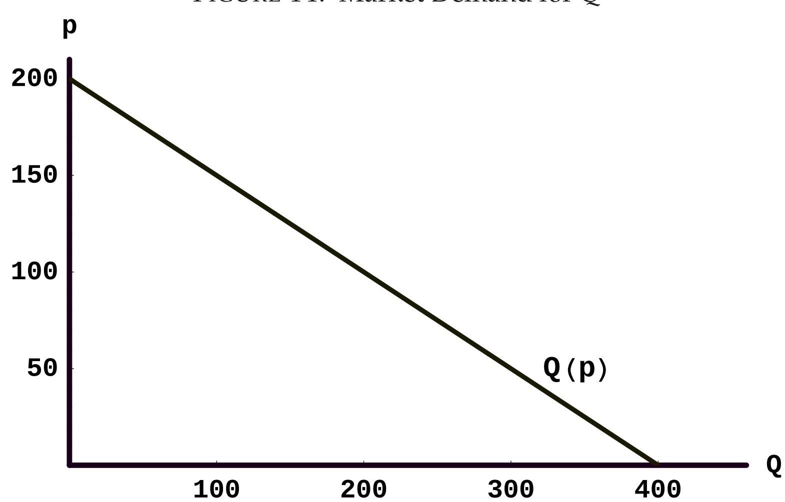


FIGURE 14. Market Demand for Q



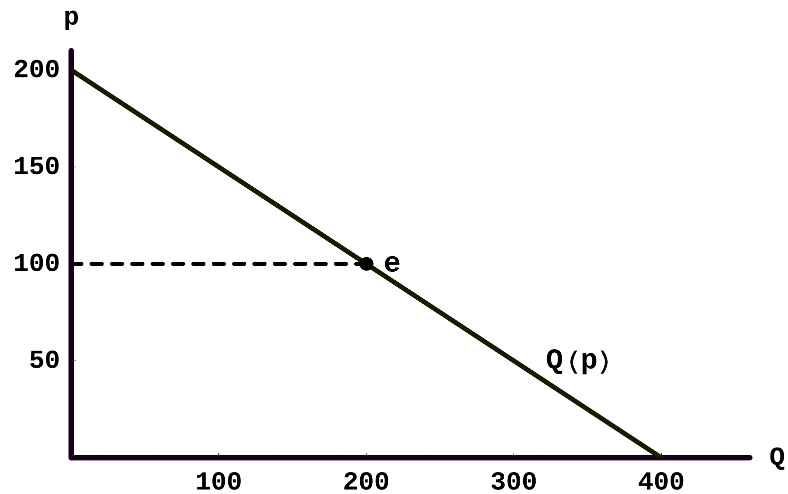
area under the inverse demand curve from 0 to the given quantity and then subtract off the revenue or expenditure at that price quantity combination.

Consider inverse demand for the aggregate demand function in equation 92

$$\begin{aligned} Q^D &= 400 \; - 2p \\ \Rightarrow \; 2p &= 400 \; - \; Q^D \\ \Rightarrow \; p^d &= 200 \; - \; \frac{1}{2} Q^D \end{aligned} \tag{93}$$

The inverse demand has a price axis intercept of 200. Now consider a price 100 and the resultant quantity of 200. Integrating we obtain

FIGURE 15. Consumer Surplus



$$
\begin{aligned}
CS(p) &= \int_{p}^{b_0} Q^D(t)\, dt \\
&= \int_{100}^{200} Q^D(p)\, dp \\
&= \int_{100}^{200} (400 - 2p)\ dp \\
&= 400\, p - p^2 \,|_{100}^{200} \\
&= \left[(400)\,(200) - 200^2 \right] - \left[(400)\,(100) - 100^2 \right] \\
&= [80000 - 40000] - [40000 - 10000] \\
&= [40000] - [30000] \\
&= 10000
\end{aligned} \tag{94}
$$

We can also obtain it by integrating inverse demand from 0 to 200 and then subtracting the expenditure of 20000.

$$\begin{aligned} TS(Q) &= \int_0^{Q^D} p\,(t)\,dt \\ &= \int_0^{200} 200 - \frac{1}{2}Q\,dQ \\ &= 200Q - \frac{1}{4}Q^2 \mid_0^{200} \\ &= \left[(200)\,(200) - \frac{1}{4}200^2 \right] - 0 \\ &= [40000] - \frac{1}{4}[40000] \\ &= [40000] - [10000] \\ &= 30000 \end{aligned} \tag{95}$$

Subtracting the expenditure of 20000 [(100)(200)] gives consumer surplus of 10000.

4.5.2. *Firm.* Consider a firm with cost function

$$cost(q) = 400 + 16q + q^2 \tag{96}$$

Of the fixed cost of \$400, \$144 is sunk (at least in the short run), and \$256 is avoidable. In the long run, all costs are avoidable.

Average cost is given by

$$AC(q) = \frac{400 + 16q + q^2}{q} \tag{97}$$

Marginal cost is given by

$$MC(q) = 16 + 2q \tag{98}$$

We first find the level of output at which average cost is minimized by setting it equal to marginal cost.

$$\begin{aligned} AC &= \frac{400 + 16q + q^2}{q} = 16 + 2q = MC_1 \\ &\Rightarrow 400 + 16q + q^2 = 16q + 2q^2 \\ &\Rightarrow 400 = q^2 \\ &\Rightarrow 20 = q \end{aligned} \tag{99}$$

The minimum level of average cost is

$$\begin{aligned} MC(20) &= 16 + 2(20) \\ &= 56 \end{aligned} \tag{100}$$

In the long run, price must be at least \$56 for the firm to continue operating. To determine the minimum price for the firm to operate in the short run we compute average avoidable cost.

$$Avoidable\ cost(q) \ = \ 256 \ + \ 16q \ + \ q^2 \tag{101}$$

and

$$AVDC(q) \ = \ \frac{256 \ + \ 16q \ + \ q^2}{q} \tag{102}$$

Average avoidable cost is minimized where it is equal to marginal cost.

$$\begin{aligned} \frac{256 \ + \ 16q \ + \ q^2}{q} &= 16 \ + \ 2q \\ \Rightarrow \ 256 \ + \ 16q \ + \ q^2 &= 16q \ + \ 2q^2 \\ \Rightarrow \ 256 &= q^2 \\ \Rightarrow \ 16 &= q \end{aligned} \tag{103}$$

The minimum level of average avoidable cost is

$$\begin{aligned} MC(16) &= 16 \ + \ 2(16) \\ &= 48 \end{aligned} \tag{104}$$

To obtain the supply function for the firm we set price equal to marginal cost.

$$\begin{aligned} MC &= 16 \ + \ 2q \ = \ p \\ \Rightarrow \ 2q &= p \ - \ 16 \Rightarrow \ q \ = \ \frac{1}{2}p \ - \ 8 \end{aligned} \tag{105}$$

This function will be the short run supply function above the minimum of average avoidable cost. This function will be the long run supply function above the minimum of average cost.

The long run supply function is

$$q^{LR} \ = \ \begin{cases} 0\,, & p \ < \ 56 \\ \frac{1}{2}p \ - \ 8, & p \ \geq \ 56 \end{cases} \tag{106}$$

The short run supply function is

$$q^{SR} \ = \ \begin{cases} 0\,, & p \ < \ 48 \\ \frac{1}{2}p \ - \ 8, & p \ \geq \ 48 \end{cases} \tag{107}$$

Now consider an industry with 12 firms of this type. The industry supply function is given by

$$\begin{aligned} Q^{SR} \ &= \ \begin{cases} 0\,, & p \ < \ 48 \\ 6p \ - \ 96, & p \ \geq \ 48 \end{cases} \\ Q^{LR} \ &= \ \begin{cases} 0\,, & p \ < \ 56 \\ 6p \ - \ 96, & p \ \geq \ 56 \end{cases} \end{aligned} \tag{108}$$

The marginal cost curve intersects the vertical axis at p = $16. The inverse industry supply or marginal cost function is

$$
\begin{aligned}
Q^S &= 6p - 96 \\
&\Rightarrow 6p = Q^S + 96 \\
&\Rightarrow p^S = \frac{1}{6}Q^S + 16
\end{aligned}
\tag{109}
$$

If the price is \$62, the typical firm will supply 23 units of output and the industry will supply 276 units of output. Producer surplus for an individual producer at a price of \$62 is given by

$$
\begin{aligned}
PS(p) &= \int_{a_0}^{p} q(t)dt \\
&= \int_{16}^{62} q(p)\,dp \\
&= \int_{16}^{62} \frac{1}{2}p - 8\,dp \\
&= \frac{1}{4}\,p^2 - 8\,p\,|_{16}^{62} \\
&= \left[\frac{1}{4}\,62^2 - (8)\,(62)\right] - \left[\frac{1}{4}\,16^2 - (8)\,(16)\right] \\
&= \left[\frac{1}{4}\,(3844) - 496\right] - \left[\frac{1}{4}\,(256) - 128\right] \\
&= [\,961 - 496\,] - [\,64 - 128\,] \\
&= [\,465\,] - [\,-64\,] \\
&= 529
\end{aligned}
\tag{110}
$$

For the industry it is given by

$$
\begin{aligned}
PSA(p) \; &= \int_{a_0}^{p} Q(t)dt \\
&= \int_{16}^{62} Q(p)\,dp \\
&= \int_{16}^{62} 6p \; - \; 96\,dp \\
&= 3p^2 \; - \; 96p \,|_{16}^{62} \\
&= \left[3\,62^2 \; - \; (96)\,(62)\,\right] \; - \; \left[3\,16^2 \; - \; (96)\,(16)\,\right] \\
&= [3\,(3844) \; - \; 5952\,] \; - \; [3,\,(256) \; - \; 1536\,] \\
&= [\,11532 \; - \; 5952\,] \; - \; [\,768 \; - \; 1536\,] \\
&= [\,5580\,] \; - \; [\,-\,768\,] \\
&= 6348
\end{aligned}
\tag{111}
$$

or

$$
\begin{aligned}
PSA(p) \; &= pQ^S \; - \; \int_0^{Q^S} p^S(t)\,dt \\
&= (62)(276) \; - \; \int_0^{276} \frac{1}{6}Q \; + \; 16\,dQ \\
&= 17112 \; - \; \frac{1}{12}Q^2 \; + \; 16Q \,|_0^{276} \\
&= 17112 \; - \; \left[\frac{1}{12}\,[\,(276)\,(276)\,] \; + \; (16)\,(276)\right] \; - \; 0 \\
&= 17112 \; - \; \frac{1}{12}\,[\,76176\,] \; - \; 4416 \\
&= 17112 \; - \; 6348 \; - \; 4416 \\
&= 6348
\end{aligned}
\tag{112}
$$

In figure 16 producer surplus is given by the area (16,62,e).

4.5.3. *Market Equilibrium.* Now combine the supply and demand equations from the previous two sections. Set supply equal to demand and compute an equilibrim price.

$$
\begin{aligned}
Q^S \; &= 6p \; - \; 96 \; = \; 400 \; - \; 2p \; = \; Q^D \\
\Rightarrow \; 8p \; &= 496 \\
\Rightarrow \; p \; &= 62 \\
\Rightarrow \; Q \; &= 276
\end{aligned}
\tag{113}
$$

This is shown in figure 17

FIGURE 16. Producer Surplus

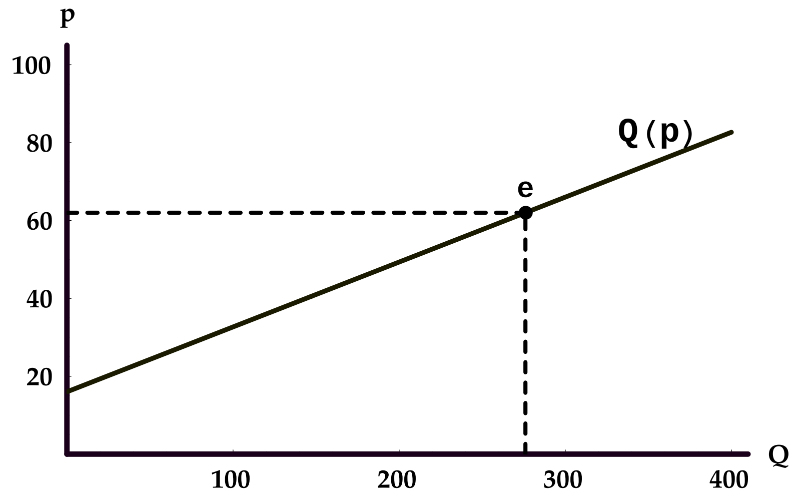
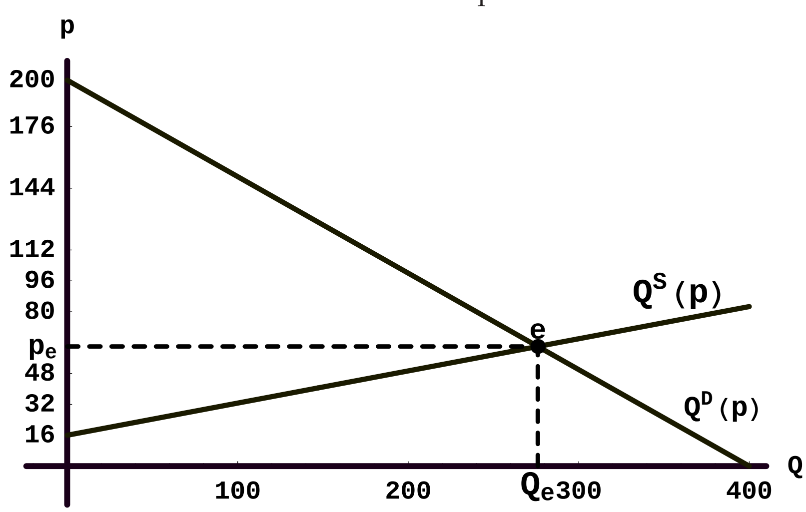


FIGURE 17. Market Equilibrium



Consumer surplus at this price is obtained by integrating the demand function from \$62 to the vertical axis intercept of \$200.

$$\begin{aligned}
CS(p) &= \int_{p}^{b_0} Q^D(t)\,dt \\
&= \int_{62}^{200} Q^D(p)\,dp \\
&= \int_{62}^{200} (400 - 2p)\,dp \\
&= 400\,p - p^2 \,|_{62}^{200} \\
&= \left[(400)(200) - 200^2 \right] - \left[(400)(62) - 62^2 \right] \\
&= [80000 - 40000] - [24800 - 3844] \\
&= [40000] - [20956] \\
&= 19044
\end{aligned} \tag{114}$$

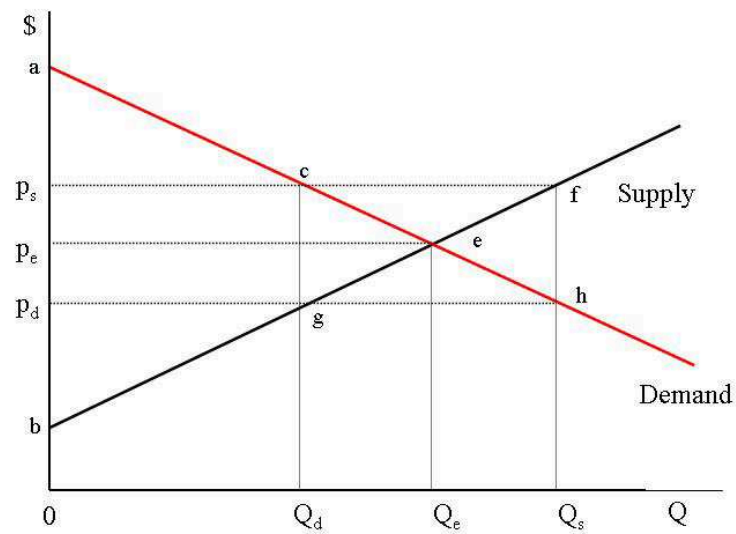
Total surplus is given by integrating the area between the demand and supply curves between \$16 and \$200. This gives

$$\begin{aligned}
TS(p) &= \int_{a_0}^{b_0} p^N(t)\,dt, \quad p^N = p^D - p^S \\
&= \int_{0}^{276} \left[\left(200 - \frac{1}{2}Q\right) - \left(\frac{1}{6}Q + 16\right)\right] dQ \\
&= \int_{0}^{276} \left[184 - \frac{2}{3}Q \right] dQ \\
&= 184Q - \frac{1}{3}Q^2 \,|_{0}^{276} \\
&= \left[(184)(276) - \frac{1}{3}276^2 \right] - 0 \\
&= 50784 - \frac{1}{3}(76176) \\
&= 50784 - 25392 \\
&= 25392
\end{aligned} \tag{115}$$

This is also the sum of producer and consumer surplus or 19044 + 6348 = 25392.

4.5.4. *Efficiency of Competition.* By using the concepts of producer and consumer surplus, we can see how competition maximizes the sum of producer and consumer surplus, or social welfare, in the quasilinear model. In figure 17we can see that the total surplus is the area below demand and above supply from the vertical axis to the quantity consumed. The maximum amount that a consumer is willing to pay for the last unit purchased in equilibrium is just the equilibrium price, p_e. The maximum amount the consumer is willing to pay for the first, the second, the third and so on up to the q_eth unit is *greater* than p_e. The areas associated with market demand and supply curves measure the welfare of all consumers and firms. The total surplus is then the area between the curves or (16,200,e). In figure 18 it is clear that the area between the curves is maximized at the competitive market price. At quantities less than q_e the area between the curves will be less. For example at the quantity q_d, the price p_s will clear the market. Consumer surplus is given by the area $p_s$ac. Producer surplus is given by the area bgcp_s. There is a net loss to society of cge. If we extend the quantity beyond q_e to q_s, consumer surplus rises to ap_dh. However producer surplus at the price p_d is the area bgp_d minus the area gfh. Total surplus is now aeb - feh or a loss compared to competition of feh. The gain to consumers of $p_e$ehp_d is less than the producer loss of $p_e$ehp_d + feh. The benefit is not worth the added cost.

FIGURE 18. Competition Maximizes Social Welfare



As an example of a program that leads to lower welfare, consider a **price support** set at p_s in figure 19, such that the government makes up the difference between what consumers will pay and the guaranteed price. Consumers will gain the amount $p_e$ebp_c, producers will gain the amount $p_s$aep_e and the government must pay $p_s$abp_c. This leads to a net loss of aeb.

4.6. **Some Comments on Partial Equilibrium.** The approach to partial equilibrium we have adopted has been based on a quasilinear model. How the competitive equilibrium you would still find it in the same way. However, it is critical for the welfare results. Without quasilinear utility, the area under the Walrasian demand curves doesn't mean anything so we will need a different welfare measure. Further, with wealth effects, welfare will depend on the distribution of the numeraire, not just the consumption good.

FIGURE 19. Social Welfare with Price Support