FUNCTIONAL FORMS IN CONSUMER THEORY

1. COBB-DOUGLAS UTILITY AND LOG-LINEAR DEMAND SYSTEMS

Consider a utility function given by

$$u = v(x) = \prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots \tag{1}$$

We assume that $\alpha_i > 0$. We sometimes assume that $\Sigma_{k=1}^{n} \alpha_k = 1$. If we maximize utility subject to a budget constraint we obtain

$$\mathcal{L} = \prod_{i=1}^{n} x_i^{\alpha_i} - \lambda \left[\Sigma_{j=1}^{n} p_j \, x_j - m \right] \tag{2}$$

Differentiating equation 2 we obtain

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\alpha_i \prod_{i=1}^{n} x_i^{\alpha_i}}{x_i} - \lambda \, p_i = 0 \tag{3a}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = - \Sigma_{j=1}^{n} p_j \, x_j + m = 0 \tag{3b}$$

If we take the ratio of any of the first n conditions we obtain

$$\frac{\frac{\alpha_i \prod_{k=1}^{n} x_k^{\alpha_k}}{x_i}}{\frac{\alpha_j \prod_{k=1}^{n} x_k^{\alpha_k}}{x_j}} = \frac{p_i}{p_j}$$

$$\Rightarrow \frac{\alpha_i \, x_j}{\alpha_j \, x_i} = \frac{p_i}{p_j} \tag{4}$$

We can now solve the equation for the jth quantity as a function of the ith quantity and the ith and jth prices. Doing so we obtain

$$\begin{aligned} x_j &= \frac{\alpha_j \, x_i \, p_i}{\alpha_i \, p_j} \\ &= \frac{\alpha_j \, x_1 \, p_1}{\alpha_1 \, p_j} \end{aligned} \tag{5}$$

where we treat the first good asymmetrically and solve for each demand as a function of the first. Now substituting in equation 3b we obtain

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\Sigma_{j=1}^{n} p_j \, x_j + m = 0$$

$$\Rightarrow \Sigma_{j=1}^{n} p_j \, \frac{\alpha_j \, x_1 \, p_1}{\alpha_1 \, p_j} = m$$

$$\Rightarrow \frac{p_1 \, x_1}{\alpha_1} \, \Sigma_{j=1}^{n} \alpha_j = m \tag{6}$$

$$\Rightarrow x_1 = \frac{\alpha_1}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{m}{p_1}$$

Similarly for the other x_k so that we have

$$x_k \, (p, \, m) = \frac{\alpha_k}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{m}{p_k} \tag{7}$$

This demand equation is clearly homogeneous of degree zero in prices and income. Also note that demand for the kth good only depends on the kth price. Also note that it is linear in income. This implies that the expenditure elasticity is equal to 1. This can be seen as follows

$$x_k = \frac{\alpha_k}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{m}{p_k}$$

$$\frac{\partial \, x_k}{\partial m} \, \frac{m}{x_k} = \left[\frac{\alpha_k}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{1}{p_k} \right] \left[\frac{m}{\frac{\alpha_k}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{m}{p_k}} \right] = 1 \tag{8}$$

We can obtain the indirect utility function by substituting the optimal x_i's in the direct utility function.

$$\psi = v \, (x \, (p, \, m)) = \prod_{i=1}^{n} x_i^{\alpha_i}$$

$$= \prod_{i=1}^{n} \left[\frac{\alpha_i}{\Sigma_{j=1}^{n} \alpha_j} \, \frac{m}{p_i} \right]^{\alpha_i}$$

$$= \prod_{i=1}^{n} \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\alpha_i} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \tag{9}$$

$$= \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\Sigma_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

We can also compute the partial derivatives of v(x) with respect to income and price. First with respect to income

$$\psi = \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\Sigma_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

$$\frac{\partial \psi}{\partial m} = \left(\Sigma_{k=1}^{n} \alpha_k \right) m^{\left(\Sigma_{k=1}^{n} \alpha_k - 1 \right)} \left[\frac{1}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\left[\Sigma_{k=1}^{n} \alpha_k \right]} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

$$= m^{\left(\Sigma_{k=1}^{n} \alpha_k - 1 \right)} \left(\Sigma_{k=1}^{n} \alpha_k \right) \left[\Sigma_{j=1}^{n} \alpha_j \right]^{\left[-\Sigma_{k=1}^{n} \alpha_k \right]} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

$$= m^{\left(\Sigma_{k=1}^{n} \alpha_k - 1 \right)} \left[\Sigma_{j=1}^{n} \alpha_j \right]^{\left[1 - \Sigma_{k=1}^{n} \alpha_k \right]} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

(10)

Then we can compute the derivative with respect to the ℓth price. This yields

$$\psi = \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\Sigma_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \alpha_i{}^{\alpha_i} p_i{}^{-\alpha_i}$$

$$\frac{\partial \psi}{\partial p_\ell} = \frac{-\alpha_\ell}{p_\ell} \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\Sigma_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}$$

(11)

If we take the ratio of 11 to 10 we obtain

$$\frac{\frac{\partial \psi}{\partial p_\ell}}{\frac{\partial \psi}{\partial m}} = \frac{\left\{ \left[\frac{-\alpha_\ell}{p_\ell} \right] \left[\frac{m}{\Sigma_{j=1}^{n} \alpha_j} \right]^{\Sigma_{k=1}^{n} \alpha_k} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \right\}}{m^{\left(\Sigma_{k=1}^{n} \alpha_k - 1 \right)} \left[\Sigma_{j=1}^{n} \alpha_j \right]^{\left[1 - \Sigma_{k=1}^{n} \alpha_k \right]} \prod_{i=1}^{n} \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i}}$$

$$= \frac{\frac{-\alpha_\ell}{p_\ell} \left[m^{\Sigma_{k=1}^{n} \alpha_k} \right] \left[\Sigma_{j=1}^{n} \alpha_j \right]^{-\Sigma_{k=1}^{n} \alpha_k}}{m^{\left(\Sigma_{k=1}^{n} \alpha_k - 1 \right)} \left[\Sigma_{j=1}^{n} \alpha_j \right]^{\left[1 - \Sigma_{k=1}^{n} \alpha_k \right]}}$$

(12)

$$= \frac{-\alpha_\ell}{p_\ell} m \frac{1}{\Sigma_{k=1}^{n} \alpha_k}$$

which is the negative of 7. This then validates Roy's identity for Cobb-Douglas preferences.

Now consider the cost minimization problem for Cobb-Douglas preferences. First set up the Lagrangian problem

$$\mathcal{L} = \Sigma_{i=1}^{n} p_i x_i - \lambda \left(\prod_{i=1}^{n} x_i^{\alpha_i} - u \right)$$

(13)

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \left[\alpha_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots \right] = 0, \quad i = 1, \ldots, n$$

(14a)

$$= p_i - \frac{\alpha_i y}{x_i} \lambda = 0, \quad i = 1, \ldots, n$$

(14b)

$$\frac{\partial \mathcal{L}}{\partial \lambda} = - \prod_{i=1}^{n} x_i^{\alpha_i} + u = 0$$

(14c)

Taking the ratio of the ith and jth equations we obtain

$$\frac{p_i}{p_j} = \frac{\alpha_i\, x_j}{\alpha_j\, x_i} \tag{15}$$

We can now solve the equation for the jth quantity as a function of the ith quantity and the ith and jth prices. Doing so we obtain

$$
\begin{aligned}
x_j &= \frac{\alpha_j\, x_i\, p_i}{\alpha_i\, p_j} \\
&= \frac{\alpha_j\, x_1\, p_1}{\alpha_1\, p_j}
\end{aligned}
\tag{16}
$$

where we treat the first good asymmetrically and solve for each demand for a good as a function of the first. Now substituting in the utility function we obtain

$$
\begin{aligned}
v &= \prod_{j=1}^{n} x_j^{\alpha_j} \\
&= \prod_{j=1}^{n} \left(\frac{\alpha_j\, x_1\, p_1}{\alpha_1\, p_j} \right)^{\alpha_j}
\end{aligned}
\tag{17}
$$

Because x_1, p_1 and α_1 do not change with j, they can be factored out of the product to obtain

$$u = \left(\frac{x_1\, p_1}{\alpha_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \tag{18}$$

We then solve this expression for x_1 as a function of u and the other x's. To do so we divide both sides by the product term to obtain

$$x_1^{\Sigma_{j=1}^{n} \alpha_j} \left(\frac{p_1}{\alpha_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} = \frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \tag{19}$$

We now multiply both sides by $\left(\frac{\alpha_1}{p_1} \right)^{\Sigma_{j=1}^{n} \alpha_j}$ to obtain

$$x_1^{\Sigma_{j=1}^{n} \alpha_j} = \frac{\left(\frac{\alpha_1}{p_1} \right)^{\Sigma_{j=1}^{n} \alpha_j} u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \tag{20}$$

If we now raise both sides to the power $\frac{1}{\Sigma_{j=1}^{n} \alpha_j}$ we find the value of x_1

$$x_1 = \left(\frac{\alpha_1}{p_1} \right) \left(\frac{u}{\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}} \right)^{\frac{1}{\Sigma_{j=1}^{n} \alpha_j}} \tag{21}$$

Similarly for the other x_k so that we have

$$x_k = \left(\frac{\alpha_k}{p_k}\right) \left(\frac{u}{\prod_{j=1}^n \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}}\right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}$$

(22)

$$= u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\left(\frac{1}{\Sigma_{j=1}^n \alpha_j}\right)} \left[\frac{\alpha_k}{p_k}\right]$$

Now if we substitute for the ith x in the cost expression we obtain

$$C = \Sigma_{i=1}^n p_i \left(\frac{\alpha_i}{p_i}\right) \left(\frac{u}{\prod_{j=1}^n \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}}\right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}$$

$$= \left(\Sigma_{i=1}^n \alpha_i\right) \left(\frac{u}{\prod_{j=1}^n \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}}\right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}$$

(23)

$$= \left(\Sigma_{i=1}^n \alpha_i\right) u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}$$

Now take the derivative of 23 with respect to p_k

$$C = \left(\Sigma_{i=1}^n \alpha_i\right) u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}$$

$$\frac{\partial C}{\partial p_k} = \left(\frac{1}{\Sigma_{j=1}^n \alpha_j}\right) \left(\Sigma_{i=1}^n \alpha_i\right) u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\left(\frac{1}{\Sigma_{j=1}^n \alpha_j} - 1\right)} \left[\frac{\alpha_k}{p_k}\right] \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)$$ (24)

$$= u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)^{\left(\frac{1}{\Sigma_{j=1}^n \alpha_j}\right)} \left[\frac{\alpha_k}{p_k}\right]$$

This is the same as 22 which verifies Shephard's lemma.

If we take the Marshallian demand from 7, and substitute in cost from 23 for m, we obtain

$$
\begin{aligned}
x_k &= \frac{\alpha_k}{\Sigma_{j=1}^n \alpha_j} \frac{m}{p_k} \\[2ex]
&= \frac{\alpha_k}{\Sigma_{j=1}^n \alpha_j} \frac{1}{p_k} \left(\Sigma_{i=1}^n \alpha_i \right) u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \\[2ex]
&= \frac{\alpha_k}{p_k} u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}}
\end{aligned}
\tag{25}
$$

which is also the same as 22. We can also substitute the indirect utility function from 9 into the Hicksian demands in 23 for u to obtain the Marshallian demands.

$$
\begin{aligned}
x_k &= \frac{\alpha_k}{p_k} u^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \\[2ex]
&= \frac{\alpha_k}{p_k} \left\{ \left[\frac{m}{\Sigma_{j=1}^n \alpha_j} \right]^{\Sigma_{k=1}^n \alpha_k} \prod_{i=1}^n \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \right\}^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \\[2ex]
&= \frac{\alpha_k}{p_k} \left\{ \left[\frac{m}{\Sigma_{j=1}^n \alpha_j} \right]^{\Sigma_{k=1}^n \alpha_k} \right\}^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left\{ \prod_{i=1}^n \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \right\}^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right)^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \\[2ex]
&= \frac{\alpha_k}{p_k} \frac{m}{\Sigma_{j=1}^n \alpha_j} \left\{ \left(\prod_{i=1}^n \left[\frac{\alpha_i}{p_i} \right]^{\alpha_i} \right) \left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j} \right)^{\alpha_j} \right) \right\}^{\frac{1}{\Sigma_{j=1}^n \alpha_j}} \\[2ex]
&= \frac{\alpha_k}{p_k} \frac{m}{\Sigma_{j=1}^n \alpha_j}
\end{aligned}
\tag{26}
$$

which is the same as 7.

2. CES UTILITY

The CES utility function is given by

$$
\begin{aligned}
u &= v(x) = - \Sigma_{k=1}^n \alpha_k x_k^\gamma, \ \alpha_k > 0, \ \gamma < 0 \\
u &= v(x) = \Sigma_{k=1}^n \alpha_k x_k^\gamma, \ \alpha_k > 0, \ 0 < \gamma < 1
\end{aligned}
\tag{27}
$$

Demand functions corresponding to the CES utility function are given by

$$
x_k(p, m) = \frac{\left(\frac{p_k}{\alpha_k} \right)^{\frac{1}{\gamma-1}} m}{\Sigma_{j=1}^n p_j \left(\frac{p_j}{\alpha_j} \right)^{\frac{1}{\gamma-1}}}
\tag{28}
$$

Note that we can write this as

$$x_k\left(p,\ m\right)\ =\ \frac{g_k\left(p\right)m}{g(p)}$$

$$where$$

$$g_k\left(p\right)\ =\ \left(\frac{p_k}{\alpha_k}\right)^{\frac{1}{\gamma-1}} \tag{29}$$

$$and$$

$$g\left(p\right)\ =\ \Sigma_{j=1}^n\ p_j\ \left(\frac{p_j}{\alpha_j}\right)^{\frac{1}{\gamma-1}}$$

As with the Cobb-Douglas utility function, demand is proportional to income or expenditure. The indirect utility function is of the form

$$\psi\ =\ v\left(x\left(p,\ m\right)\right)\ =\ \frac{m}{\left[\ \Sigma_{j=1}^n\ \alpha_j^{\frac{-1}{\gamma-1}}\ p_j^{\frac{\gamma}{\gamma-1}}\ \right]^{\frac{\gamma-1}{\gamma}}} \tag{30}$$

3. KLEIN-RUBIN (STONE-GEARY) UTILITY AND THE LINEAR EXPENDITURE SYSTEM

3.1. **Log form of the utility function and Marshallian demand functions.** Consider a utility function given by

$$u = v(x) = \Sigma_{k=1}^n a_k \, log \, [x_k - b_k], \, a_i > 0, \, (x_i - b_i) > 0, \, \Sigma_{k=1}^n a_k = 1 \tag{31}$$

If we maximize utility subject to a budget constraint we obtain

$$\mathcal{L} = \Sigma_{k=1}^n a_k \, log \, [x_k - b_k] - \lambda[\Sigma_{j=1}^n p_j \, x_j - m] \tag{32a}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{a_i}{x_i - b_i} - \lambda p_i = 0, \quad i = 1, 2, \cdots, n \tag{32b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\Sigma_{j=1}^n p_j \, x_j + m = 0 \tag{32c}$$

If we take the ratio any of the first n conditions we obtain

$$\frac{\frac{a_i}{x_i - b_i}}{\frac{a_j}{x_j - b_j}} = \frac{p_i}{p_j}$$

$$\Rightarrow \frac{a_i \, (x_j - b_j)}{a_j \, (x_i - b_i)} = \frac{p_i}{p_j}$$

$$\Rightarrow a_i x_j - a_i b_j = \frac{a_j \, p_i \, (x_i - b_i)}{p_j} \tag{33}$$

$$\Rightarrow x_j = \frac{a_j \, p_i \, (x_i - b_i)}{a_i \, p_j} + b_j$$

$$\Rightarrow x_j = \frac{a_j \, p_1 \, (x_1 - b_1)}{a_1 \, p_j} + b_j$$

where we treat the first good asymmetrically and solve for each demand as a function of the first. Now substituting in equation 32c we obtain

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\Sigma_{j=1}^n p_j \, x_j + m = 0$$

$$\Rightarrow \Sigma_{j=1}^n p_j \left[\frac{a_j \, p_1 \, (x_1 - b_1)}{a_1 \, p_j} + b_j \right] = m$$

$$\Rightarrow \frac{p_1}{a_1} \Sigma_{j=1}^n \left[\frac{a_j \, p_1 \, x_1 - a_j \, p_1 \, b_1}{a_1} \right] + \Sigma_{j=1}^n p_j \, b_j = m$$

$$\Rightarrow \frac{p_1}{a_1} \Sigma_{j=1}^n a_j \, x_1 - a_j \, b_1 + \Sigma_{j=1}^n p_j \, b_j = m$$

$$\Rightarrow \frac{p_1}{a_1} \Sigma_{j=1}^n a_j \, (x_1 - b_1) + \Sigma_{j=1}^n p_j \, b_j = m \tag{34}$$

$$\Rightarrow \frac{p_1}{a_1} \, (x_1 - b_1) \Sigma_{j=1}^n a_j + \Sigma_{j=1}^n p_j \, b_j = m$$

$$\Rightarrow \frac{p_1}{a_1} \, x_1 - \frac{p_1}{a_1} \, b_1 + \Sigma_{j=1}^n p_j \, b_j = m$$

$$\Rightarrow \frac{p_1}{a_1} \, x_1 = m + \frac{p_1}{a_1} \, b_1 - \Sigma_{j=1}^n p_j \, b_j$$

$$\Rightarrow x_1 = \frac{a_1}{p_1} \, m + b_1 - \frac{a_1}{p_1} \Sigma_{j=1}^n p_j \, b_j$$

Similarly for the other x_k so that we have

$$x_k(p, m) = \frac{a_k}{p_k} m + b_k - \frac{a_k}{p_k} \Sigma_{j=1}^n p_j b_j \tag{35}$$

If we directly exploit the fact $\Sigma_{k=1}^n a_k = 1$, we can solve for the demand functions in a slightly simpler manner. Rearrange the first order conditions to obtain

$$\frac{a_i}{x_i - b_i} = \lambda p_i, \quad i = 1, 2, \cdots, n$$
$$\Rightarrow a_i = \lambda p_i (x_i - b_i) \tag{36}$$

Now sum over i to obtain

$$\Sigma_{i=1}^n a_i = \Sigma_{i=1}^n \lambda p_i (x_i - b_i)$$
$$\Rightarrow 1 = \lambda (\Sigma_{i=1}^n p_i x_i - \Sigma_{i=1}^n p_i b_i)$$
$$\Rightarrow \lambda = \frac{1}{\Sigma_{i=1}^n p_i x_i - \Sigma_{i=1}^n p_i b_i} \tag{37}$$
$$\Rightarrow \lambda = \frac{1}{m - \Sigma_{i=1}^n p_i b_i}$$

Equation 32c allows us to substitute m for $\Sigma_{i=1}^n p_i x_i$. If we substitute 37 back into 36 and rearrange we obtain

$$\frac{a_i}{x_i - b_i} = \lambda p_i$$
$$\Rightarrow (x_i - b_i) = \frac{a_i}{\lambda p_i}$$
$$\Rightarrow x_i = b_i + \frac{a_i}{p_i \left[\frac{1}{m - \Sigma_{j=1}^n p_j b_j} \right]} \tag{38}$$
$$= b_i + \frac{a_i}{p_i} \left[m - \Sigma_{j=1}^n p_j b_j \right]$$

We often write it in expenditure form as

$$p_k x_k = p_k b_k + a_k \left(m - \Sigma_{j=1}^n p_j b_j \right) \tag{39}$$

This demand system is called the linear expenditure system (LES) because expenditure in linear in prices and income. The demand equations are clearly homogeneous of degree zero in prices and income. Unlike the Cobb-Douglas function, demand depends on all prices.

The LES has the following interpretation. The consumer spends a "committed" amount on each good ($b_1, b_2, ... b_n$) and then divides the remaining or "supernumerary" expenditure $\left(m - \Sigma_{j=1}^n p_j x_j \right)$ among the goods in a fixed proportions ($a_1, a_2, ... a_n$). The quantities ($b_1, b_2, ... b_n$) are often called a necessary consumption basket. Notice that the derivative of 35 with respect to expenditure (income) is a constant.

3.2. **Product form of the utility function, the cost function and Hicksian demand functions.** The cost and indirect utility functions for the LES are usually obtained assuming the direct utility function is a transformation (which doesn't matter) of the function given in 31, that is

$$u = v(x) = \prod_{k=1}^{n} [x_k - b_k]^{a_k}, \quad a_i > 0, \quad (x_i - b_i) > 0, \quad \Sigma_{k=1}^{n} a_k = 1 \tag{40}$$

First set up the Lagrangian problem

$$\mathcal{L} = \Sigma_{k=1}^{n} p_k x_k - \lambda \left(\prod_{k=1}^{n} (x_k - b_k)^{a_k} - u^0 \right) \tag{41}$$

The first order conditions are as follows

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \left[a_i (x_1 - b_1)^{a_1} (x_2 - b_2)^{a_2} \dots (x_{i-1} - b_{i-1})^{a_{i-1}} (x_i - b_i)^{a_i-1} (x_{i+1} - b_{i+1})^{a_{i+1}} \dots \right] = 0, \quad i = 1, \dots, n \tag{42a}$$

$$= p_i - \frac{a_i v}{x_i - b_i} \lambda = 0, \quad i = 1, \dots, n \tag{42b}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = - \prod_{k=1}^{n} (x_k - b_k)^{a_k} + u = 0 \tag{42c}$$

Taking the ratio of the ith and jth equations we obtain

$$\frac{p_i}{p_j} = \frac{a_i (x_j - b_j)}{a_j (x_i - b_i)} \tag{43}$$

We can now solve the equation for the jth quantity as a function of the ith quantity and the ith and jth prices. Doing so we obtain

$$x_j = \frac{a_j p_i (x_i - b_i)}{a_i p_j} + b_j \tag{44}$$

Now treat the first good asymmetrically and solve for each demand for a good as a function of the first to obtain

$$x_j = \frac{a_j p_1 (x_1 - b_1)}{a_1 p_j} + b_j \tag{45}$$

Now substituting in the utility function we obtain

$$v = \prod_{k=1}^{n} (x_j - b_k)^{a_k}$$
$$= \prod_{k=1}^{n} \left(\frac{a_k p_1 (x_1 - b_1)}{a_1 p_k} \right)^{a_k} \tag{46}$$

Because x_1, p_1, a_1, and b_1 do not change with k, they can be factored out of the product to obtain

$$u = \left(\frac{(x_1 - b_1) p_1}{a_1} \right)^{\Sigma_{k=1}^{n} a_k} \prod_{k=1}^{n} \left(\frac{a_k}{p_k} \right)^{a_k} \tag{47}$$

We then solve this expression for x_1 as a function of u and the other x's. To do so we divide both sides by the product term to obtain

$$(x_1 - b_1)^{\Sigma_{k=1}^n a_k} \left(\frac{p_1}{a_1}\right)^{\Sigma_{k=1}^n a_k} = \frac{u}{\prod_{k=1}^n \left(\frac{a_k}{p_k}\right)^{a_k}} \tag{48}$$

We now multiply both sides by $\left(\frac{a_1}{p_1}\right)^{\Sigma_{j=1}^n a_j}$ to obtain

$$(x_1 - b_1)^{\Sigma_{k=1}^n a_k} = \frac{\left(\frac{a_1}{p_1}\right)^{\Sigma_{k=1}^n a_k} u}{\prod_{k=1}^n \left(\frac{a_k}{p_k}\right)^{a_k}} \tag{49}$$

If we now raise both sides to the power $\frac{1}{\Sigma_{j=1}^n a_j}$ we find the value of x_1

$$x_1 - b_1 = \left(\frac{a_1}{p_1}\right) \left(\frac{u}{\prod_{k=1}^n \left(\frac{a_k}{p_k}\right)^{a_k}}\right)^{\frac{1}{\Sigma_{k=1}^n a_k}} \tag{50}$$

Similarly for the other x_j so that we have

$$x_j - b_j = \left(\frac{a_j}{p_j}\right) \left(\frac{u}{\prod_{k=1}^n \left(\frac{a_k}{p_k}\right)^{a_k}}\right)^{\frac{1}{\Sigma_{k=1}^n a_k}}$$

$$= u^{\frac{1}{\Sigma_{k=1}^n a_k}} \left(\prod_{k=1}^n \left(\frac{p_k}{a_k}\right)^{a_k}\right)^{\left(\frac{1}{\Sigma_{k=1}^n a_k}\right)} \left[\frac{a_j}{p_j}\right] \tag{51}$$

$$\Rightarrow x_j = u^{\frac{1}{\Sigma_{k=1}^n a_k}} \left(\prod_{k=1}^n \left(\frac{p_k}{a_k}\right)^{a_k}\right)^{\left(\frac{1}{\Sigma_{k=1}^n a_k}\right)} \left[\frac{a_j}{p_j}\right] + b_j$$

Now note that $\Sigma_{k=1}^n a_k = 1$ by assumption so that we obtain

$$x_j = u \left[\frac{a_j}{p_j}\right] \left(\prod_{k=1}^n \left(\frac{p_k}{a_k}\right)^{a_k}\right) + b_j \tag{52}$$

Now if we substitute for the ith x in the cost expression we obtain

$$C = \Sigma_{i=1}^n \, p_i \left(u \, \left[\frac{a_i}{p_i} \right] \left(\prod_{k=1}^n \left(\frac{p_k}{a_k} \right)^{a_k} \right) \right) + \Sigma_{i=1}^n \, p_i b_i$$

$$= u \left(\prod_{k=1}^n \left(\frac{p_k}{a_k} \right)^{a_k} \right) \Sigma_{i=1}^n a_i + \Sigma_{k=1}^n \, p_k b_k$$

$$= u \left(\prod_{k=1}^n \left(\frac{p_k}{a_k} \right)^{a_k} \right) + \Sigma_{k=1}^n \, p_k b_k \tag{53}$$

$$= u \left(\prod_{k=1}^n a_k^{-a_k} \right) \left(\prod_{k=1}^n p_k^{a_k} \right) + \Sigma_{k=1}^n \, p_k b_k$$

We usually subsume the constant term $\left(\prod_{k=1}^n a_k^{-a_k} \right)$ somewhere in the function and write the cost function for the LES as

$$C(u, \, p) = \Sigma_{k=1}^n \, p_k \, b_k + u \prod_{k=1}^n p_k^{a_k} \tag{54}$$

The indirect utility function for the LES is given by

$$\psi(m, \, p) = \frac{m - \Sigma_{k=1}^n \, p_k \, b_k}{\left(\prod_{k=1}^n \, p_k^{a_k} \right) \left(\prod_{k=1}^n \, a_k^{-a_k} \right)} \tag{55}$$

or

$$\psi(m, \, p) = \frac{m - \Sigma_{j=1}^n \, p_j \, b_j}{\prod_{j=1}^n \, p_j^{a_j}} \tag{56}$$

4. DEMAND SYSTEMS LINEAR IN EXPENDITURE

Demand systems linear in expenditure are of the form

$$x_i(m, \, p) = c_i(p) + b_i(p) \, m \tag{57}$$

In a classic paper Gorman (1961) showed any demand system that is consistent with utility maximization and linear in expenditure must be of the form

$$x_i(m, \, p) = f_i(p) - \frac{g_i(p)}{g(p)} \, f(p) + \frac{g_i(p)}{g(p)} \, m$$

$$= f_i(p) + \frac{g_i(p)}{g(p)} \, [m - f(p)] \tag{58}$$

where both f(p) and g(p) are homogeneous of degree one, $g_i = \frac{\partial g}{\partial x_i}$ and $f_i = \frac{\partial f}{\partial x_i}$. If linearity is required for all price-expenditure combinations in which p and m are strictly positive, then this results in expenditure proportionality because the budget constraint implies that

$$\Sigma_{k=1}^{n} \, p_k \, x_k \; = \; \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; + \; \frac{g_k \, (p)}{g \, (p)} \, [\, m \; - \; f \, (p) \,] \, \right] \; = m$$

$$\Rightarrow \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; - \; \left[\frac{f \, (p) \, g_k \, (p)}{g \, (p)} \right] \, \right] \; + \; \Sigma_{k=1}^{n} \, p_k \left[\, \left[\frac{g_k \, (p)}{g \, (p)} \right] \, m \, \right] \; = m$$

$$\Rightarrow \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; - \; \left[\frac{f \, (p) \, g_k \, (p)}{g \, (p)} \right] \, \right] \; + \; \frac{m}{g \, (p)} \, \Sigma_{k=1}^{n} \, p_k \, g_k \, (p) \; = m$$

$$\Rightarrow \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; - \; \left[\frac{f \, (p) \, g_k \, (p)}{g \, (p)} \right] \, \right] \; + \; \frac{m}{g \, (p)} \, g \, (p) \; = m, \quad \text{by homogeneity}$$

$$\Rightarrow \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; - \; \left[\frac{f \, (p) \, g_k \, (p)}{g \, (p)} \right] \, \right] \; + \; m \; = m$$

$$\Rightarrow \Sigma_{k=1}^{n} \, p_k \left[\, f_k \, (p) \; - \; \left[\frac{f \, (p) \, g_k \, (p)}{g \, (p)} \right] \, \right] \; = 0$$

(59)

and nonnegativity of consumption near zero expenditure implies $f_j \, - \, \frac{f \, g_j}{g} \; \geq \; 0$. This then implies that $f_j \, - \, \frac{f \, g_j}{g} \; = \; 0$ for all j. Such a demand system comes from an indirect utility function of the form

$$\psi \, (p \, , \, m) \; = \; \frac{m \; - \; f \, (p)}{g \, (p)} \tag{60}$$

To see this take the derivatives of 60 with respect to the kth price and income.

$$\frac{\partial \, \psi \, (p \, , \, m)}{\partial \, p_k} \; = \; \frac{g \, (p) \, (- \, f_k \, (p) \,) \; + \; (\, f \, (p) \; - \; m \,) \, g_k \, (p)}{g \, (p)^2}$$

$$= \; \frac{f \, (p) \, g_k \, (p) \; - \; g \, (p) \, f_k \, (p) \; - \; m \, g_k \, (p)}{g \, (p)^2} \tag{61}$$

$$\frac{\partial \, \psi \, (p \, , \, m)}{\partial \, m} \; = \; \frac{1}{g \, (p)}$$

Then take the ratio of the derivatives and simplify

$$\frac{\frac{\partial \, \psi \, (p \, , \, m)}{\partial \, p_k}}{\frac{\partial \, \psi \, (p \, , \, m)}{\partial \, m}} \; = \; \frac{f \, (p) \, g_k \, (p) \; - \; g \, (p) \, f_k \, (p) \; - \; m \, g_k \, (p)}{g \, (p)}$$

$$= \; - \, f_k \, (p) \; + \; \frac{f \, (p) \, g_k \, (p)}{g \, (p)} \; - \; \frac{g_k \, (p)}{g \, (p)} \, m \tag{62}$$

$$= \; - \, x_k \, (p \, , \, m)$$

by Roy's identity. The form of the indirect utility function in 60 is known as the "Gorman polar form". As an example consider $f(p) \; = \; \Sigma_{k=1}^{n} p_k \, b_k$ and $g(p) \; = \; \Pi_{k=1}^{n} p_k^{a_k}$ with $\Sigma_{k=1}^{n} \, a_k \; = \; 1$.

Then $f_k \; = \; b_k$, $g_k = \frac{a_k \, g(p)}{p_k}$ and $\frac{g_k}{g(p)} \; = \; \frac{a_k}{p_k}$. This gives as a demand system

$$x_i \, (m \, , \, p) \; = \; f_i \, (p) \; - \; \frac{g_i \, (p)}{g \, (p)} \, f \, (p) \; + \; \frac{g_i \, (p)}{g \, (p)} \, m$$

$$= b_i \; - \; \frac{a_i}{p_i} \, \Sigma_{j=1}^{n} \, p_j \, b_j \; + \; \frac{a_i}{p_i} \, m \tag{63}$$

which is the linear expenditure system. While the complete class of utility functions leading to demand systems linear in expenditure is not fully characterized, one important class leading to linear systems is

$$v(x) = T[h(x - b)] \tag{64}$$

where $T'(\cdot) > 0$ and h is linearly homogeneous. A function v that satisfies equation 64 is said to be homothetic to the point b. The indifference curves are scaled up versions of a base indifference curve and the expansion path radiates in a straight line from a translated origin. The Gorman polar form corresponding to equation 64 has $g(\cdot)$ dual to $h(\cdot)$ with $f(p) = \Sigma_{k=1}^{n} p_k b_k$. In this sense the LES system is a translation of the Cobb-Douglas system. We can similarly define translations of CES systems.

5. DEMAND SYSTEMS QUADRATIC IN EXPENDITURE

Demand systems quadratic in expenditure are of the form

$$x_i(m, p) = c_i(p) + b_i(p) m + a_i(p) m^2 \tag{65}$$

This specification allows for non-linear relationships between income and consumption as is more commonly thought to occur. Van Daal and Merkies commenting on a paper by Hoew, Pollak and Wales show that theoretically consistent demand system that are quadratic in expenditure must be of the form

$$\begin{aligned} x_i(m, p) =& \frac{1}{\gamma}\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right) m^2 + \left[\frac{g_i}{g} + \frac{2f}{\gamma}\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right)\right] m + \\ & \frac{f^2}{\gamma}\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right) - \frac{g_i}{g} f + f_i + \chi\left(\frac{g}{\gamma}\right)\left(\frac{g^2}{\gamma}\right)\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right) \end{aligned} \tag{66}$$

We can also write this as

$$x_i(m, p) = \frac{1}{\gamma}\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right)(m - f)^2 + \frac{g_i}{g}(m - f) + f_i + \chi\left(\frac{g}{\gamma}\right)\left(\frac{g^2}{\gamma}\right)\left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right) \tag{67}$$

In 66 and 67 we assume that f, g, and γ are homogeneous of degree 1 and that the function $\chi(\cdot)$ is a function of one variable. An alternative is to let $\alpha(p) = \frac{g(p)^2}{\gamma(p)}$ and then write 67 as

$$x_i(m, p) = \frac{1}{g}^2\left(\alpha_i - \frac{g_i}{g}\alpha\right)(m - f)^2 + \frac{g_i}{g}(m - f) + f_i + \chi\left(\frac{\alpha}{g}\right)\left(\alpha_i - \frac{g_i}{g}\alpha\right) \tag{68}$$

When $\gamma(\cdot) = 0$, these quadratic systems are characterized by the indirect utility functions

$$\psi(p, m) = -\frac{g(p)}{m - f(p)} - \frac{g(p)}{\gamma(p)} \tag{69}$$

or

$$\psi(p, m) = -\frac{g(p)}{m - f(p)} - \frac{\alpha(p)}{g(p)} \tag{70}$$

If we write the Gorman polar form in equivalent form as

$$\psi(p, m) = \frac{-g(p)}{m - f(p)} \tag{71}$$

it is clear that the quadratic system just adds a term that is homogeneous of degree zero to the Gorman polar form indirect utility function.

A couple of forms for these functions have been proposed in the literature. They both use the same functions for f(p) and g(p) as the linear expenditure system, that is, $f(p) = \Sigma_{k=1}^{n} p_k b_k$ and $g(p) = \Pi_{k=1}^{n} p_k^{a_k}$ with $\Sigma_{k=1}^{n} a_k = 1$. The first specifies α(p) as $\alpha(p) = \frac{\lambda g(p)^2}{\Pi_{k=1}^{n} p_k^{x_k}}$, $\Sigma_k c_k = 1$ where parameters λ and c_k are added. The second adds only n parameters and chooses $\alpha(p) = \Sigma_{k=1}^{n} p_k c_k$. With α(p) = 0, these reduce to the LES.

6. DEMAND SYSTEMS CHARACTERIZED BY PRICE INDEPENDENT GENERALIZED LINEARITY (MUELLBAUER)

6.1. **Two term polynomial demand systems and PIGL.** Consider a demand system with two terms, one linear and one with a power of m. We can write this as

$$x_i(m, p) = b_i(p) m + d_i(p) m^\sigma \tag{72}$$

where σ might be 1 or 2 or some higher power. This form does not have a term that is independent of income. Muellbauer has shown that theoretically plausible demand systems of this form must have the structure

$$x_i(m, p) = \left(\frac{g_i}{g} - \frac{\gamma_i}{\gamma}\right) \frac{m^\sigma}{\gamma^{\sigma-1}} + \frac{g_i}{g} m \tag{73}$$

where both g(p) and γ(p) are linearly homogeneous. The indirect utility function consistent with these systems is

$$\psi(p, m) = \frac{-m^{1-\sigma} - \gamma(p)^{1-\sigma}}{g(p)^{1-\sigma}} \tag{74}$$

One problem with such a system is that there no allowance for expenditure that is not dependent on income. Empirical work tends to find Engel curves that do not pass through the origin and so there is some desire to add a c(p) term to such systems. The idea is to modify the indirect utility function by subtracting a function of price from the income term. Specifically, if the indirect utility function is given by

$$\psi(p, m) = \Omega(p, m) \tag{75}$$

we transform it to be

$$\hat{\psi}(p, m) = \Omega(p, m - f(p)) \tag{76}$$

The idea is that we reduce available income for expenditure by some function of prices f(p). If we apply Roy's identity to the original and transformed indirect utility functions we obtain

$$x_k\left(p,\,m\right)\;=\;\frac{-\,\dfrac{\partial\,\psi\,(p\;m)}{\partial\,p_k}}{\dfrac{\partial\,\psi\,(p\;m)}{\partial\,m}}\;=\;\frac{-\,\dfrac{\partial\,\Omega}{\partial\,p_k}}{\dfrac{\partial\,\Omega}{\partial\,m}}$$

$$\hat{x}_k\left(p,\,m\right)\;=\;\frac{-\,\dfrac{\partial\,\hat{\psi}\,(p\;m)}{\partial\,p_k}}{\dfrac{\partial\,\hat{\psi}\,(p\;m)}{\partial\,m}}\;=\;\frac{-\left[\dfrac{\partial\,\Omega}{\partial\,p_k}\;-\;\dfrac{\partial\,\Omega}{\partial\,m}\,f_i\left(p\right)\right]}{\dfrac{\partial\,\Omega}{\partial\,m}}\tag{77}$$

$$=\;\frac{-\,\dfrac{\partial\,\Omega}{\partial\,p_k}}{\dfrac{\partial\,\Omega}{\partial\,m}}\;+\;f_i\left(p\right)$$

$$=\;x_k\left(p,\,m\,-\,f\left(p\right)\right)\;+\;f_i\left(p\right)$$

What this says is that we can add a constant term (not depending on income) to the demand function by subtracting some function of price from the measure of income and then adding back the derivative of that function as the constant. Consider a couple of examples. First consider the simple proportional system

$$x_i\left(m,\,p\right)\;=\;b_i\left(p\right)\,m\tag{78}$$

The modified system is given by

$$\hat{x}_i\left(m,\,p\right)\;=\;b_i\left(p\right)\left(m\,-\,f\left(p\right)\right)\;+\;f_i\left(p\right)\tag{79}$$

This is the same as the Gorman polar form in 57 where $b_i(p) = g_i(p)/g(p)$. Next consider the PIGL system in 72 where $\sigma = 2$. We obtain

$$\psi\left(p,\,m\right)\;=\;\frac{-\,m^{-1}\,-\,\gamma\left(p\right)^{-1}}{g\left(p\right)^{-1}}\;=\;\frac{-\,m^{-1}}{g\left(p\right)^{-1}}\,-\,\frac{\gamma\left(p\right)^{-1}}{g\left(p\right)^{-1}}\;=\;-\,\frac{g}{m}\,-\,\frac{g}{\gamma}\tag{80}$$

Now transform it by introducing f(p). We obtain

$$\hat{\psi}\left(p,\,m\right)\;=\;-\,\frac{g}{m\,-\,f\left(p\right)}\,-\,\frac{g}{\gamma}\tag{81}$$

which is the same as the quadratic system in 69. This could also be done for a PIGL system 73 where $\sigma = 3$. This would give

$$x_i\left(m,\,p\right)\;=\;\left(\frac{g_i}{g}\,-\,\frac{\gamma_i}{\gamma}\right)\frac{m^3}{\gamma^2}\,+\,\frac{g_i}{g}\,m$$

$$\hat{x}_i\left(m,\,p\right)\;=\;\left(\frac{g_i}{g}\,-\,\frac{\gamma_i}{\gamma}\right)\frac{\left(m\,-\,f\left(p\right)\right)^3}{\gamma^2}\,+\,\frac{g_i}{g}\left(m\,-\,f\left(p\right)\right)\,+\,f_i\left(p\right)\tag{82}$$

When expanded this will give a system with linear, quadratic and cubic terms in expenditure and a term independent of expenditure. But this system is not completely general in that quadratic and cubic terms will be proportional. The same is true of all polynomial demand systems.

6.2. **PIGLOG demand systems.** Muellbauer also introduced a class of demand systems related to the polynomial PIGL class. The PIGLOG class of demand systems has the form

$$x_i(m, p) = b_i(p) m + d_i(p) m \log[m] \tag{83}$$

Expenditure thus enters linearly and as a log function of m. In share form equation 83 can be written

$$
\begin{aligned}
\omega_i(m, p) = \frac{p_i x_i}{m} &= \frac{p_i b_i(p) m + p_i d_i(p) m \log[m]}{m} \\
&= \frac{p_i b_i(p) m}{m} + \frac{p_i d_i(p) m \log[m]}{m} \\
&= p_i b_i(p) + p_i d_i(p) \log[m] \\
&= \hat{b}_i(p) + \hat{d}_i(p) \log[m]
\end{aligned}
\tag{84}
$$

Thus the share form has a term that is independent of income along with a term that is linear in log m. Muellbauer has shown that theoretically plausible systems of this form must be written as

$$x_i(m, p) = \frac{g_i}{g} m - \frac{G_i(p)}{G(p)} [\log[m] - \log[g(p)]] m \tag{85}$$

Here G(p) is homogeneous of degree zero and g(p) is homogeneous of degree 1 and the subscript i as usual denotes a derivative with respect to the ith price. The indirect utility function associated with 85 is

$$\hat{\psi}(p, m) = G(p) [\log[m] - \log[g(p)]] \tag{86}$$

Roy's identity holds as is obvious from

$$
\begin{aligned}
x_k(p, m) &= \frac{-\frac{\partial \psi(p, m)}{\partial p_k}}{\frac{\partial \psi(p, m)}{\partial m}} \\
\\
&= -\frac{G(P) \left(\frac{-g_i(p)}{g(p)}\right) + G_i(p) [\log[m] - \log[g(p)]]}{\frac{G(p)}{m}} \\
&= m \left(\frac{g_i(p)}{g(p)}\right) - \frac{G_i(p)}{G(p)} [\log[m] - \log[g(p)]] m
\end{aligned}
\tag{87}
$$

We will show later that some forms of the translog and the AIDS demand systems are special cases of the PIGLOG system. The class can be extended by adding terms that are quadratic in log m. More general systems are discussed in a series of papers by Lewbel in the late 1980's and early 1990's.

7. LINEAR TRANSLOG DEMAND SYSTEM

This is usually written in share form as

$$\omega_i(m, p) = \frac{p_i x_i(m, p)}{m} = \frac{p_i b_i}{m} + \left[\alpha_i + \Sigma_{j=1}^n \beta_{ij} \log[p_j]\right] \left[1 - \Sigma_{k=1}^n b_k \frac{p_k}{m}\right] \tag{88}$$

$$\beta_{ij} = \beta_{ji}, \ \Sigma_{i=1}^n \beta_{ij} = 0, \ \Sigma_{k=1}^n \alpha_k = 0$$

The summation condition on the β's assures homogeneity of degree zero in prices and income because $\log[\lambda\ p_j\ ] = \log[\lambda] + \log[p_j]$ and the sum in the brackets can be divided into two terms, one equal to $\log[\lambda]\ \Sigma_{j=1}^{n}\ \beta_{ij}$.

8. Log Translog

This is also written in share form as

$$\omega_i(m,\ p)\ =\ \frac{\alpha_i\ +\ \Sigma_{j=1}^{n}\ \beta_{ij}\ log\ [p_j]\ -\ log[m]\ \Sigma_{j=1}^{n}\ \beta_{ij}}{1\ +\ \Sigma_{j=1}^{n}\ \Sigma_{k=1}^{n}\ \beta_{kj}\ log\ [p_j]} \tag{89}$$

$$\beta_{ij}\ =\ \beta_{ji},\ \Sigma_{k=1}^{n}\Sigma_{j=1}^{n}\beta_{kj}\ =\ 0,\quad \Sigma_{k=1}^{n}\alpha_k\ =\ 1$$

9. Almost Ideal Demand System (AIDS)

$$\omega_i(m,\ p)\ =\ \alpha_i\ +\ \Sigma_{k=1}^{n}\ \beta_{ki}\ log\ [p_k]\ +\ \gamma\ [\log\ [m]\ -\ \log\ [g\ (p)]] \tag{90a}$$

$$\log\ [g\ (p)]\ = \alpha_0\ +\ \Sigma_{k=1}^{n}\ \alpha_k\ log\ [p_k]\ +\ \frac{1}{2}\ \Sigma_{k=1}^{n}\ \Sigma_{j=1}^{n}\ \beta_{k\,j}\ log\ [p_k]\ log\ [p_j] \tag{90b}$$

$$\beta_{ij}\ =\ \beta_{ji},\ \Sigma_{k=1}^{n}\beta_k\ =\ 0,\quad \Sigma_{k=1}^{n}\alpha_k\ =\ 1\,,\ \Sigma_{k=1}^{n}\gamma_k\ =\ 0$$

We assume that g(p) is homogeneous of degree 1 in p which means that $g(\lambda p) = \lambda g(p)$ and $\log[g(\lambda p)] = \log[\lambda] + \log[g(p)]$.

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