PROFIT FUNCTIONS

1. REPRESENTATION OF TECHNOLOGY

1.1. **Technology Sets.** The technology set for a given production process is defined as

$$T = \{(x, y): \; x \,\epsilon\, R^n_+, \; y \,\epsilon\, R^m_+ : x \text{ can produce } y\}$$

where x is a vector of inputs and y is a vector of outputs. The set consists of those combinations of x and y such that y can be produced from the given x.

1.2. **The Output Correspondence and the Output Set.**

1.2.1. *Definitions.* It is often convenient to define a production correspondence and the associated output set.

1: The output correspondence P, maps inputs x ϵ R^n_+ into subsets of outputs, i.e., P: $R^n_+ \rightarrow 2^{R^m_+}$. A correspondence is different from a function in that a given domain is mapped into a set as compared to a single real variable (or number) as in a function.

2: The output set for a given technology, P(x), is the set of all output vectors y ϵ R^m_+ that are obtainable from the input vector x ϵ R^n_+. P(x) is then the set of all output vectors y ϵ R^m_+ that are obtainable from the input vector x ϵ R^n_+. We often write P(x) for both the set based on a particular value of x, and the rule (correspondence) that assigns a set to each vector x.

1.2.2. *Relationship between P(x) and T(x,y).*

$$P\,(x) \;=\; (y \,:\, (x,\, y\,)\; \epsilon\; T)$$

1.2.3. *Properties of P(x).*

P.1a: P.1 No Free Lunch. 0 ϵ P(x) $\forall$ x ϵ R^n_+.
P.1b: y $\notin$ P(0), y > 0.
P.2: Input Disposability. $\forall$ x ϵ R^n_+ , P(x) $\subseteq$ P(θx), $\theta \geq 1$.
P.2.S: Strong Input Disposability. $\forall$ x, x′ ϵ R^n_+ , x′ $\geq$ x $\Rightarrow$ P(x) $\subseteq$ P(x′).
P.3: Output Disposability. $\forall$ x ϵ R^n_+ , y ϵ P(x) and $0 \leq \lambda \leq 1 \Rightarrow \lambda$y ϵ P(x).
P.3.S: Strong Output Disposability. $\forall$ x ϵ R^n_+ , y ϵ P(x) $\Rightarrow$ y′ ϵ P(x), $0 \leq$ y′ $\leq$ y.
P.4: Boundedness. P(x) is bounded for all x ϵ $R^n{}_+$.
P.5: T(x) is a closed set P: $R^n_+ \rightarrow 2^{R^m_+}$ is a closed correspondence, i.e., if [$x^\ell \rightarrow x^0$, $y^\ell \rightarrow y^0$ and y^ℓ ϵ P(x^ℓ), $\forall$ ℓ] then y^0 ϵ P(x^0).
P.6: Attainability. If y ϵ P(x), y $\geq$ 0 and x $\geq$ 0, then $\forall$ $\theta \geq 0$, $\exists$ $\lambda_\theta \geq 0$ such that θy ϵ P(λ_θx).
P.7: P(x) is convex
P(x) is convex for all x ϵ R^n_+ .
P.8: P is quasi-concave.
The correspondence P is quasi-concave on R^n_+ which means $\forall$ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, P(x) $\cap$ P(x′) $\subseteq$ P(θx + (1-θ)x′)
P.9: Convexity of T(x). P is concave on R^n_+ which means $\forall$ x, x′ ϵ R^n_+ , $0 \leq \theta \leq 1$, θP(x)+(1-θ)P(x′) $\subseteq$ P(θx + (1-θ)x′)

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1.2.4. *Properties of T(x,y).*

T.1a: Inaction and No Free Lunch. (0,y) ϵ T $\forall$ x ϵ R^n_+ and y ϵ R^m_+. This implies that T(x,y) is a non-empty subset of R^{m+n}_+

T.1b: (0,y) $\notin$ T, y $\geqq$ 0, y $\neq$ 0.

T.2: Input Disposability. If (x,y) ϵ T and $\theta \geq 1$ then (θx, y) ϵ T.

T.2.S: Strong Input Disposability. If (x,y) ϵ T and x′ $\geqq$ x, then (x′, y) ϵ T.

T.3: Output Disposability. $\forall$ (x,y) ϵ R^{n+m}_+ ,if (x, y) ϵ T and $0 < \lambda \leqq 1$ then (x, λy) ϵ T.

T.3.S: Strong Output Disposability. If (x,y) ϵ T and y′ $\leqq$ y, then (x, y′) ϵ T.

T.4: Boundedness. For every finite input vector x $\geqq$ 0, the set y ϵ P(x) is bounded from above. This implies that only finite amounts of output can be produced from finite amounts of inputs.

T.5: T(x) is a closed set. The assumption that P(x) and V(y) are closed does not imply that T is a closed set, so it is assumed. Specifically, if [$x^\ell \rightarrow x^0$, $y^\ell \rightarrow y^0$) and (x^ℓ, y^ℓ) ϵ T, $\forall$ ℓ] then (x^0 y^0) ϵ T.

T.9: T is a convex set. This is not implied by the convexity of P(x) and V(y). Specifically, a technology could exhibit a diminishing rate of technical substitution, a diminishing rate of product transformation and still exhibit increasing returns to scale.

2. Profit Maximization and the Profit Function

2.1. **Profit Maximization.**

2.1.1. *Setup of problem.* The general firm-level maximization problem can be written in a number of alternative ways.

$$\pi \;=\; \max_{x\;y}\,[\sum_{j=1}^{m} p_j\; y_j \;-\; \sum_{i=1}^{n} w_i\, x_i], \quad \text{such that (x, y)} \in \text{T}. \tag{1}$$

where T is represents the graph of the technology or the technology set. The problem can also be written as

$$\pi \;=\; \max_{x\;y}\,[\sum_{j=1}^{m} p_j y_j \;-\; \sum_{i=1}^{n} w_i\, x_i] \text{ such hat x} \in \text{V(y)]} \tag{2a}$$

$$\pi \;=\; \max_{x\;y}\,[\sum_{j=1}^{m} p_j y_j \;-\; \sum_{i=1}^{n} w_i\, x_i] \text{ such that y} \in \text{P(x)]} \tag{2b}$$

where the technology is represented by V(y), the input requirement set, or P(x),the output set. Though T is non-empty, closed and convex, profit may not attain a maximum on T, i.e. profit can be unbounded even when the technology set is well behaved if output price is higher than input price. Consider for example the production function f(x)= x + $x^{1/2}$. The production function is concave but profit will go to infinity if p > w. Therefore we often write

$$\pi \;=\; \sup_{x\;y}\,[\sum_{j=1}^{m} p_j\; y_j \;-\; \sum_{i=1}^{n} w_i\, x_i], \quad \text{such that (x, y)} \in \text{T}. \tag{3}$$

where sup stands for supremum and could be infinity. We understand that in all practical problems the sup is a max.

If we carry out the maximization in equation 1 or equation 3, we obtain a vector of optimal outputs and a vector of optimal outputs such that y is producible given x and profits cannot be increased. We denote these optimal input and output choices as y(p,w) and x(p,w) where it is implicit that y and x are vectors.

2.1.2. *One output and two input example.* In the case of a single output (such that f(x) is defined) and two inputs we obtain the following.

$$\pi \;=\; p\, f\,(x_1,\, x_2\,) \;-\; w_1\, x_1 \;-\; w_2\, x_2 \tag{4}$$

If we differentiate the expression in equation 4 with respect to each input we obtain

$$\begin{aligned} \frac{\partial\, \pi}{\partial\, x_1} &= p\, \frac{\partial f\,(x_1,\, x_2)}{\partial x_1} \;-\; w_1 \;=\; 0 \\ \frac{\partial\, \pi}{\partial\, x_2} &= p\, \frac{\partial f\,(x_1,\, x_2)}{\partial x_2} \;-\; w_2 \;=\; 0 \end{aligned} \tag{5}$$

If we solve the equations in 5 for x_1 and x_2, we obtain the optimal values of x for a given p and w. As a function of w for a fixed p, this gives the vector of factor demands for x.

$$\begin{aligned} x^* &= x(p,\, w_1,\, w_2) \\ &= (\, x_1(p,\, w_1,\, w_2),\; x_2(p,\, w_1,\, w_2)\,) \end{aligned} \tag{6}$$

The optimal output is given by

$$y^* \;=\; f\;(\, x_1(p,\, w_1,\, w_2),\; x_2(p,\, w_1,\, w_2)\,) \tag{7}$$

2.2. **The Profit Function.** If we substitute the optimal input demand from equation 6 into equation 1 or equation 4, we obtain the profit function. The profit function is usually designated by π.

$$\begin{aligned} \pi(p, w) \;&=\; \sum_{j=1}^{m} p_j\; y_j(p, w) \;-\; \sum_{i=1}^{n} w_i\, x_i(p, w) \\ &= p\, f(x_1(p, w),\, x_2(p, w)) \;-\; w_1\, x_1(p, w) \;-\; w_2\, x_2(p, w) \end{aligned} \tag{8}$$

Notice that π is a function of p and w, not x or y. The optimal x and optimal y have already been chosen. The function tells us what profits will be (assuming the firm is maximizing profits) given a set of output and input prices.

To help understand how π(p,w) only depends on p and w, consider the profit function for the case of two inputs.

$$\pi(p, w_1,\, w_2) \;=\; p\, f(x_1(p, w_1,\, w_2),\, x_2(p, w_1,\, w_2)) \;-\; w_1\, x_1(p, w_1,\, w_2) \;-\; w_2\, x_2(p, w_2,\, w_2) \tag{9}$$

Consider the derivative of π(p,w) with respect to p.

$$\begin{aligned} \frac{\partial \pi(p, w)}{\partial p} &= p\, \frac{\partial f(x_1(p, w),\, x_2(p, w))}{\partial x_1}\, \frac{\partial x_1(p, w)}{\partial p} \;+\; p\, \frac{\partial f(x_1(p, w),\, x_2(p, w))}{\partial x_2}\, \frac{\partial x_2(p, w)}{\partial p} \\ &\quad + f(x_1(p, w),\, x_2(p, w)) \;-\; w_1\, \frac{\partial x_1(p, w))}{\partial p} \;-\; w_2\, \frac{\partial x_2(p, w))}{\partial p} \end{aligned} \tag{10}$$

Now collect terms containing $\frac{\partial x_1(p,w)}{\partial p}$ and $\frac{\partial x_2(p,w)}{\partial p}$.

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial p} &= \frac{\partial x_1(p,w)}{\partial p}\left[p\,\frac{\partial f(x_1(p,w),\, x_2(p,w))}{\partial x_1} \;-\; w_1\right] \\ &+ \frac{\partial x_2(p,w))}{\partial p}\,\left[p\,\frac{\partial f(x_1(p,w),\, x_2(p,w))}{\partial x_2} \;-\; w_2\right] \\ &+ f(x_1(p,w),\, x_2(p,w))\end{aligned} \tag{11}$$

But the first order conditions in equation 5 imply that

$$p\,\frac{\partial f(x_1\, x_2)}{\partial x_i} \;-\; w_i \;=\; 0, \qquad i = 1, 2 \tag{12}$$

This means that the two bracketed terms in equation 11 are equal to zero so that

$$\frac{\partial \pi(p,w)}{\partial p} \;=\; f(x_1(p,w),\, x_2(p,w)) \tag{13}$$

which depends only on p and w.

3. Properties of, or Conditions on, the Profit Function

3.1. $\pi.1$. π(p, w) is an extended real valued function (it can take on the value of $+\infty$ for finite prices) defined for all (p, w) $\geqq$ (0_m, 0_n)) and π(p, w) $\geq$ pa - wb for a fixed vector (a, b) $\geq$ (0_m, 0_n). This implies that π(p, w) $\geqq$ 0 if (0_m, 0_n) $\in$ T(x,y),which we normally assume.

3.2. $\pi.2$. π is nonincreasing in w

3.3. $\pi.3$. π is nondecreasing in p

3.4. $\pi.4$. π is a convex function

3.5. $\pi.5$. π is homogeneous of degree 1 in p and w.

3.6. $\pi.6$ **Hotelling's Lemma.**

$$\frac{\partial \pi(p,\, w)}{\partial p_j} = y_j(p,\, w) \tag{14a}$$

$$\frac{\partial \pi(p,\, w)}{\partial w_i} \;=\; -\, x_i(p,\, w) \tag{14b}$$

4. Discussion of Properties of the Profit Function

For ease of exposition, consider the case where there is a single output. This is easily generalized by replacing f(x(p,w)) with y(p,w) and letting p be a vector.

4.1. $\pi.1$. π(p, w) is an extended real valued function (it can take on the value of $+\infty$ for finite prices) defined for all (p, w) $\geqq$ (0_m, 0_n)) and π(p, w) $\geq$ pa - wb for a fixed vector (a, b) $\geq$ (0_m, 0_n). This implies that π(p, w) $\geqq$ 0.

The profit function can be infinity due to the fact that maximum profits may be infinity as discussed in section 2.1.1. π(p, w) $\geq$ pa - wb for a fixed vector (a, b) because π(p, w) is the maximum profit at output prices p and input prices w. Any other input combination is bounded by π(p, w).

4.2. π.2. π is nonincreasing in w

Let optimal input be $\tilde{x}$(p,w) with prices p and $\tilde{w}$ and $\hat{x}$(p,w) with prices p and $\hat{w}$. Now assume that $\hat{w} > \tilde{w}$. It is clear that pf($\hat{x}$(p,w)) - $\tilde{w}$ $\hat{x}$(p,w) $\leq$ pf($\tilde{x}$(p,w)) - $\tilde{w}$ $\tilde{x}$(p,w) because the optimal x with $\tilde{w}$ is $\tilde{x}$. However, pf($\hat{x}$) - $\hat{w}$ $\hat{x}$ < pf($\hat{x}$) - $\tilde{w}$ $\hat{x}$ because $\hat{w} > \tilde{w}$ by assumption. So we obtain

$$\begin{aligned} & pf(\hat{x}(p,w)\,) \,-\, \hat{w}\,\hat{x}(p,w) \,<pf(\hat{x}(p,w)) \,-\, \tilde{w}\,\hat{x}(p,w) \,\leq\, pf(\tilde{x}(p,w)\,) - \tilde{w}\,(p,w)\,\tilde{x}(p,w) \\ \Rightarrow\, & pf(\hat{x}(p,w)\,) \,-\, \hat{w}\,\hat{x}(p,w) \,\leq\, pf(\tilde{x}(p,w)\,) - \tilde{w}\,\tilde{x}(p,w) \end{aligned} \tag{15}$$

4.3. π.3. π is nondecreasing in p

Let profit be $\tilde{\pi}$ ($\tilde{p}$,w) with prices $\tilde{p}$ and w and $\hat{\pi}$($\hat{p}$,w) with prices $\hat{p}$ and w. Now assume that $\hat{p} > \tilde{p}$. It is clear that

$$\hat{p}f(\tilde{x}(p,w)\,) \,-\, w\,\tilde{x}(p,w) \,\leq\, \hat{p}\,f(\hat{x}(p,w)) \,-\, w\,\hat{x}(p,w) \tag{16}$$

because $\hat{x}(p,w)$ is optimal for prices $\hat{p}$. However,

$$\tilde{p}f(\tilde{x}(p,w)\,) \,-\, w\,\tilde{x}(p,w) \,<\, \hat{p}\,f(\tilde{x}(p,w)) \,-\, w\,\tilde{x}(p,w) \tag{17}$$

because $\hat{p} > \tilde{p}$ by assumption. So we obtain,

$$\begin{aligned} & \tilde{p}f(\tilde{x}(p,w)\,) \,-\, w\,\tilde{x}(p,w) \,<\, \hat{p}\,f(\tilde{x}(p,w)) \,-\, w\,\tilde{x}(p,w) \,\leq\, \hat{p}\,f(\hat{x}(p,w)) \,-\, w\,\hat{x}(p,w) \\ \Rightarrow\, & \tilde{p}f(\tilde{x}(p,w)\,) \,-\, w\,\tilde{x}(p,w) \,\leq\, \hat{p}\,f(\hat{x}(p,w)) \,-\, w\,\hat{x}(p,w) \end{aligned} \tag{18}$$

4.4. π.4. π is a convex function

Consider the definition of a convex function. A function f: $R^{m+n} \to R^1$ is said to be convex if $f\,(\,\lambda\,x_1 \,+\, (1\,-\,\lambda)\,x_2) \,\leq\, \lambda\,f\,(x_1) \,+\, (1\,-\,\lambda\,)f\,(\,x_2)$ for each vector x_1, $x_2 \in R^{m+n}$ and for each $\lambda \in$ [0, 1]. In the case of the profit function π maps an n+m vector containing output and input prices into the real line. Alternatively, a differentiable function π is convex in R_+^{m+n} if and only if

$$\begin{aligned} \pi(p,w) \,\geq\, \pi\,(\bar{p},\,\bar{w}\,) \,+\, \nabla_p\,\pi\,(\bar{p},\,\bar{w}\,)'\,(p\,-\bar{p}) \,+\, \nabla_w\,\pi\,(\bar{p},\,\bar{w}\,)'\,(w\,-\bar{w}), \\ for\ each\ distinct\ (p,w),(\bar{p},\,\bar{w}) \,\in\, R_+^{m+n}. \end{aligned} \tag{19}$$

Let (y,x) be the profit maximizing choices of y and x when prices are(p, w) and let (y′,x′) be profit maximizing maximizing choices of y and x when prices are(p′, w′). Let (p″ w″) be a linear combination of the price vectors (p,w) and (p′,w′), i.e.

$$\begin{aligned} (p'',\,w'') =&\lambda\,\times\,(p,w) + (1-\lambda)\,\times\,(p',w') \\ =&\,[\,\lambda\,p\,+\,(1-\lambda)\,p',\,\lambda\,w\,+\,(1-\lambda)\,w'\,] \end{aligned} \tag{20}$$

Then let (y″,x″) be the profit maximizing choices of y and x when prices are(p″, w″). This then implies that

$$\pi(p'',\,w'') \,=\, p''\,y'' \,-\, w''\,x'' \tag{21}$$

Now substitute the definition of p″ and w″ from equation 20 into 21

$$\begin{aligned} \pi(p'', w'') &= p''\, y'' \;-\; w''\, x'' \\ &= [\, \lambda\, p \;+\; (1-\lambda)\, p' \,]\; y'' \;-\; [\, \lambda\, w \;+\; (1-\lambda)\, w' \,]\; x'' \\ &= \lambda\, p\, y'' \;-\; \lambda\, w\, x'' \;+\; (1-\lambda)\, p'\, y'' \;-\; (1-\lambda)\, w'\, x'' \\ &= \lambda\; [\, p\, y'' \;-\; w\, x''] \;+\; (1-\lambda)\; [\, p'\, y'' \;-\; w'\, x''] \end{aligned} \tag{22}$$

Be the definition of profit maximization, we know that when prices are (p,w) any output input combination other than (y,x) will yield lower profits. Similarly when prices are (p′, w′). We then can write

$$\begin{aligned} \lambda\; [\, p\, y'' \;-\; w\, x''] \;\leq\; \lambda\; [\, p\, y \;-\; w\, x] \;=\; \lambda\, \pi\, (p,\, w) \\ (1-\lambda)\; [\, p'\, y'' \;-\; w'\, x''] \leq\; (1-\lambda)\; [\, p'\, y' \;-\; w'\, x'] \;=\; (1-\lambda\,)\pi\, (p',\, w') \end{aligned} \tag{23}$$

Now write equation 23 without the middle terms

$$\begin{aligned} \lambda\; [\, p\, y'' \;-\; w\, x''] \;\leq\; \lambda\, \pi\, (p,\, w) \\ (1-\lambda)\; [\, p'\, y'' \;-\; w'\, x''] \leq\; (1-\lambda\,)\pi\, (p',\, w') \end{aligned} \tag{24}$$

Now add the two inequalities in equation 24 to obtain

$$\lambda\; [\, p\, y'' - w\, x''] + (1-\lambda)\; [\, p'\, y'' - w'\, x''] \;\leq\; \lambda\, \pi\, (p,\, w) + (1-\lambda\,)\pi\, (p',\, w') \tag{25}$$

Now substitute for the left hand side of equation 25 from equation 22 to obtain

$$\pi(p'', w'') \;\leq\; \lambda\, \pi\, (p,\, w) + (1-\lambda\,)\pi\, (p',\, w') \tag{26}$$

Becauuse π(p,w) is convex, we know that its Hessian matrix is positive semidefinite. This means that the diagonals of the Hessian matrix are all positive or zero, i.e., $\frac{\partial^2\, \pi(p,w)}{\partial\, p_j^2} \geq 0$ and $\frac{\partial^2\, \pi(p,w)}{\partial\, w_i^2} \geq 0$.

We can visualize convexity if we hold all input prices fixed and only consider a firm with a single output, or hold all but one output price fixed. In figure 1we can see that the tangent lies below the curve. At prices (p*, w*), profits are at the level π(p*, w*). If we hold the output and input levels fixed at (y*, x*) and change p, we move along the tangent line denoted by $\tilde{\Pi}$(p*, $\bar{w}$). Profits are less along this line than along the profit function because we are not adjusting x and y to account for the change in p.

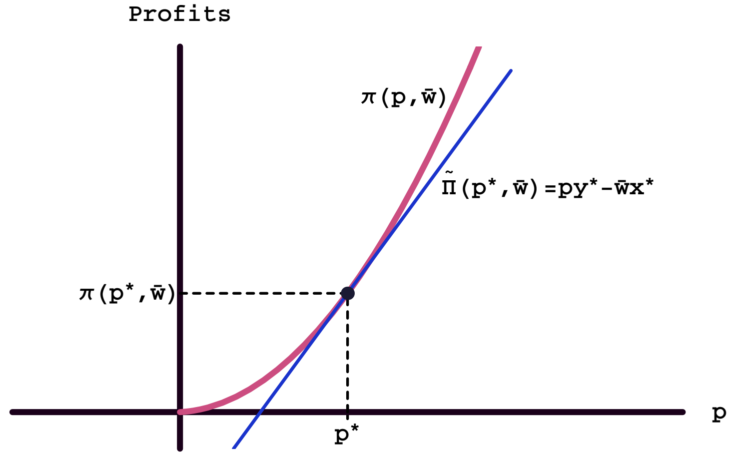
4.5. $\pi.5$. π is homogeneous of degree 1 in p and w.

Consider the profit maximization problem,

$$\pi(p, w) \;=\; \max_{x\;y}\, [\sum_{j=1}^{m}\, p_j\; y_j \;-\; \sum_{i=1}^{n}\, w_i\, x_i],\ \text{such that (x, y)} \in \text{T}. \tag{27}$$

Now multiply all prices by λ and denote the new profit mximization problem as

FIGURE 1. The Profit Function is Convex



$$
\begin{aligned}
\pi^*(p, w) &= \max_{x\ y} [\sum_{j=1}^{m} \lambda\, p_j\, y_j \;-\; \sum_{i=1}^{n} \lambda\, w_i\, x_i], \text{ such that (x, y)} \in \text{T} \\
&= \lambda \max_{x\ y} [\sum_{j=1}^{m} p_j\, y_j \;-\; \sum_{i=1}^{n} w_i\, x_i] \\
\Rightarrow\ \pi^*(p, w) &= \lambda\, \pi(p, w)
\end{aligned} \tag{28}
$$

4.6. π.6 **Hotelling's Lemma.**

$$\frac{\partial \pi(p,\, w)}{\partial p_j} = y_j(p,\, w) \tag{29a}$$

$$\frac{\partial \pi(p,\, w)}{\partial w_i} = -\, x_i(p,\, w) \tag{29b}$$

We have already shown part a of equation 29 in equation 13 for the case of a single output. We can derive part b of f equation 29 in a similar manner. Consider the derivative of π(p,w_1, w_2) with respect to w_1. The profit function is given by

$$\pi(p, w_1,\, w_2) \;=\; p\, f(x_1(p, w_1,\, w_2),\, x_2(p, w_1,\, w_2)) \;-\; w_1\, x_1(p, w_1,\, w_2) \;-\; w_2\, x_2(p, w_2,\, w_2) \tag{30}$$

Taking the derivative with respect to w_1 we obtain

$$
\begin{aligned}
\frac{\partial \pi(p, w)}{\partial w_1} &= p\, \frac{\partial f(x_1(p, w),\, x_2(p, w))}{\partial x_1}\, \frac{\partial x_1(p, w)}{\partial w_1} \;+\; p\, \frac{\partial f(x_1(p, w),\, x_2(p, w))}{\partial x_2}\, \frac{\partial x_2(p, w)}{\partial w_1} \\
&\quad -\; w_1\, \frac{\partial x_1(p, w))}{\partial w_1} \;-\; x_1(p, w) \;-\; w_2\, \frac{\partial x_2(p, w))}{\partial w1}
\end{aligned} \tag{31}
$$

Now collect terms containing $\frac{\partial x_1(p,w)}{\partial w_1}$ and $\frac{\partial x_2(p,w)}{\partial w_1}$.

$$\begin{aligned}\frac{\partial \pi(p,w)}{\partial w_1} &= \frac{\partial x_1(p,w)}{\partial w_1}\left[p\,\frac{\partial f(x_1(p,w),\,x_2(p,w))}{\partial x_1} \,-\, w_1\right] \\ &+ \frac{\partial x_2(p,w))}{\partial w_1}\left[p\,\frac{\partial f(x_1(p,w),\,x_2(p,w))}{\partial x_2} \,-\, w_2\right] \\ &- x_1(p, w_1,\, w_2)\end{aligned} \tag{32}$$

But the first order conditions in equation 5 imply that

$$p\,\frac{\partial f(x_1\, x_2)}{\partial x_i} \,-\, w_i \,=\, 0, \quad i = 1, 2 \tag{33}$$

This means that the two braketed terms in equation 32 are equal to zero so that

$$\frac{\partial \pi(p,w)}{\partial w1} \,=\, -\,x_1(p, w_1, w_2) \tag{34}$$

We can show this in a more general manner as follows. Define a function g(p,w) as follows

$$g(p,w) \,=\, \pi(p,w) \,-\, [\,py^* \,-\, wx^*\,] \tag{35}$$

where (y^*, x^*) is some production plan that is not necessarily optimal at prices (p,w). At prices (p^*, w^*), (y^*, x^*) will be the profit maximizing production. The first term in equation 35 will always be greater than or equal to the second term. Thus g(p,w) will reach its minimum when prices are (p^*, w^*). Consider then the conditions for minimizing g(p,w).

$$\begin{aligned}\frac{\partial g(p^*,w^*)}{\partial p_j} &= \frac{\partial \pi(p^*,w^*)}{\partial p_j} \,-\, y_j^* \,=\, 0, \quad j = 1, 2, \ldots, m \\ \frac{\partial g(p^*,w^*)}{\partial w_i} &= \frac{\partial \pi(p^*,w^*)}{\partial w_i} \,+\, x_j^* \,=\, 0, \quad i = 1, 2, \ldots, n\end{aligned} \tag{36}$$

Rearranging equation 36 we obtain

$$\begin{aligned}&\frac{\partial \pi(p^*,w^*)}{\partial p_j} \,-\, y_j^* \,=\, 0, \quad j = 1, 2, \ldots, m \\ &\Rightarrow \frac{\partial \pi(p^*,w^*)}{\partial p_j} = y_j^*, \quad j = 1, 2, \ldots, m \\ &\frac{\partial \pi(p^*,w^*)}{\partial w_i} \,+\, x_j^* \,=\, 0, \quad i = 1, 2, \ldots, n \\ &\Rightarrow \frac{\partial \pi(p^*,w^*)}{\partial w_i} \,=\, -\,x_i^*, \quad i = 1, 2, \ldots, m\end{aligned} \tag{37}$$

A number of important implications come from equation 37.

1: We can obtain output supply and input demand equations by differentiating the profit function.

2: If we have an expression (equation) for the profit function, we can obtain output supply and input demand equations or functional forms for such equations by differentiating the profit function as compared to solving a maximization problem. This would allow one to find functional forms for estimating supply and demand without solving maximization problems.

3: Given that the output supply and input demand functions are derivatives of the profit function, we can determine many of the properties of these response functions by understanding the properties of the profit function. For example, because the profit function is homogeneous of degree one in p and w, its derivatives are homogeneous of degree zero. This means that output supply and input demand equations are homogeneous of degree zero, i.e., multiplying all prices by the same constant will not change output supply or input demand.

4: The second derivatives of the profit function are the first derivatives of the output supply and input demand functions. Properties of the second derivatives of the profit function are properties of the first derivatives of the output supply and input demand functions.

5: The symmetry of second order cross partial derivatives leads to symmetry of first cross price derivatives of the output supply and input demand functions.

5. Numeric Example

5.1. **Production function.** Consider the following production function.

$$\begin{aligned} y &= f(x_1,\, x_2) \\ &= 24x_1 \,+\, 14x_2 \,-x_1^2 \,+\, x_1\,x_2 \,-\, x_2^2 \end{aligned} \tag{38}$$

The first and second partial derivatives are given by

$$\begin{aligned} \frac{\partial f(x_1,\, x_2)}{\partial x_1} &= 24 \,-\, 2x_1 \,+\, x_2 \\ \frac{\partial f(x_1,\, x_2)}{\partial x_2} &= 14 \,+\, x_1 \,-\, 2x_2 \\ \frac{\partial^2 f(x_1,\, x_2)}{\partial x_1^2} &= -\,2 \\ \frac{\partial^2 f(x_1,\, x_2)}{\partial x_1 \partial x_2} &= 1 \\ \frac{\partial^2 f(x_1,\, x_2)}{\partial x_2^2} &= -\,2 \end{aligned} \tag{39}$$

The Hessian is

$$\nabla^2 f(x_1,\, x_2) \,=\, \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \tag{40}$$

The determinant of the Hessian is given by

$$\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \,=\, 4 \,-\, 1 \,=\, 3 \tag{41}$$

The rate of technical substitution is given by

$$RTS \,=\, \frac{\partial x_2(y,\, x_1)}{\partial x_1} \,=\, -\frac{24 \,+\, 2x_1 \,-\, x_2}{14 \,+\, x_1 \,-\, 2x_2} \tag{42}$$

The elasticity substitution is given by

$$\begin{aligned}\sigma_{12} &= \frac{-f_1\, f_2\, (x_1\, f_1\, +\, x_2\, f_2)}{x_1\, x_2\, (f_{11}\, f_2^2\, -\, 2f_{12}\, f_1\, f_2\, +\, f_{22}\, f_1^2)} \\ &= \frac{-(14\, +\, x_1\, -\, 2x_2)(24\, -\, 2x_1\, +\, x_2)\, (x_1^2\, -\, 12x_1\, -\, 7x_2\, -\, x_1x_2\, +\, x_2^2)}{x_1\, x_2\, \left(1108\, +\, 3x_1^2\, -\, 72x_1\, -\, 42x_2\, -\, 3x_1x_2\, +\, 3x_2^2\right))}\end{aligned} \tag{43}$$

5.2. **Profit maximization.** Profit is given by

$$\begin{aligned}\pi &= p\, f(x_1,\, x_2)\, -\, w_1\, x_1\, -\, w_2\, x_2 \\ &= p\, \left[\, 24x_1\, +\, 14x_2\, -\, 2x_1^2\, +\, x_1\, x_2\, - 2x_2^2\right]\, -\, w_1\, x_1\, -\, w_2\, x_2\end{aligned} \tag{44}$$

We maximize profit by taking the derivatives of 44 setting them equal to zero and solving for x_1 and x_2.

$$\begin{aligned}\frac{\partial \pi}{\partial x_1} &= p\,[\,24\, -\, 2x_1\, +\, x_2\,]\, -\, w_1\, =\, 0 \\ \frac{\partial \pi}{\partial x_2} &= p\,[\,14\, +\, x_1\, -\, 2x_2\,]\, -\, w_2\, =\, 0\end{aligned} \tag{45}$$

Rearranging 45 we obtain

$$24\, -\, 2x_1\, +\, x_2\, =\, \frac{w_1}{p} \tag{46a}$$

$$14\, +\, x_1\, -\, 2x_2\, =\, \frac{w_2}{p} \tag{46b}$$

Now solve equation 46a for x_1 as follows

$$x_1\, =\, \frac{24p\, -\, w_1\, +\, px_2}{2p} \tag{47}$$

Then substitute x_1 from equation 47 into equation 46b as follows

$$
\begin{aligned}
14 \,+\, \left(\frac{24p \,-\, w_1 \,+\, px_2}{2p}\right) \,-\, 2x_2 \,&= \frac{w_2}{p} \\
\Rightarrow \left(\frac{28p \,+\, 24p \,-\, w_1 \,+\, px_2 \,-\, 4px_2}{2p}\right) \,&= \frac{w_2}{p} \\
\Rightarrow \left(\frac{52p \,-\, w_1 \,-\, 3px_2}{2p}\right) \,&= \frac{w_2}{p} \\
\Rightarrow \left(\frac{-3px_2}{2p}\right) \,&= \frac{w_2}{p} \,-\, \left(\frac{52p \,-\, w_1}{2p}\right) \\
&= \left(\frac{2w_2 \,-\, 52p \,+\, w_1}{2p}\right) \\
\Rightarrow -3px_2 \,&= 2w_2 \,-\, 52p \,+\, w_1 \\
\Rightarrow x_2 \,&= \frac{-2w_2 \,+\, 52p \,-\, w_1}{3p} \\
&= \frac{52p \,-\, w_1 \,-\, 2w_2}{3p}
\end{aligned}
\tag{48}
$$

If we substitute the last expression in equation 48 for x_2 in equation 47 we obtain

$$
\begin{aligned}
x_1 \,&= \frac{24p \,-\, w_1 \,+\, px_2}{2p} \\
&= \frac{24p \,-\, w_1}{2p} \,+\, \frac{p}{2p}\, x_2 \\
&= \frac{24p \,-\, w_1}{2p} \,+\, \frac{1}{2}\left(\frac{52p \,-\, w_1 \,-\, 2w_2}{3p}\right) \\
&= \frac{72p \,-\, 3w_1 \,+\, 52p \,-\, w_1 \,-\, 2w_2}{6p} \\
&= \frac{124p \,-\, 4w_1 \,-\, 2w_2}{6p} \\
&= \frac{62p \,-\, 2w_1 \,-\, w_2}{3p}
\end{aligned}
\tag{49}
$$

5.3. **Necessary and sufficient conditions for a maximum.** Consider the Hessian of the profit equation.

$$
\nabla^2 \pi(x_1,\, x_2) \;=\; p \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \;=\; \begin{pmatrix} -2p & p \\ p & -2p \end{pmatrix}
\tag{50}
$$

For a maximum we need the diagonal elements to be negative and the determinant to be positive. The diagonal elements are negative. The determinant of the Hessian is $4p^2$-p_2 = $3p^2$, which is positive.

5.4. **Optimal output.** We can obtain optimal output as a function of p, w_1 and w_2 by substituting the optimal values of x_1 and x_2 into the production function.

$$
\begin{aligned}
y(p, w_1, w_2) &= f(x_1(p, w_2, w_2),\ x_2(p, w_1, w_2)) \\
&= 24\left(\frac{62p\ -\ 2w_1\ -\ w_2}{3p}\right)\ +\ 14\left(\frac{52p\ -\ w_1\ -\ 2w_2}{3p}\right) \\
&- \left(\frac{62p\ -\ 2w_1\ -\ w_2}{3p}\right)^2 + \left(\frac{62p\ -\ 2w_1\ -\ w_2}{3p}\right)\left(\frac{52p\ -\ w_1\ -\ 2w_2}{3p}\right) \\
&\quad - \left(\frac{52p\ -\ w_1\ -\ 2w_2}{3p}\right)^2
\end{aligned}
\tag{51}
$$

Now put everything over a common denominator and multiply out as follows

$$
\begin{aligned}
y(p, w_1, w_2) &= \frac{(72p)\,(62p\ -\ 2w_1\ -\ w_2)}{9p^2} + \frac{42p\,(52p\ -\ w_1\ -\ 2w_2)}{9p^2} \\
&- \frac{(62p\ -\ 2w_1\ -\ w_2)^2}{9p^2} + \frac{(62p\ -\ 2w_1\ -\ w_2)\,(52p\ -\ w_1\ -\ 2w_2)}{9p^2} \\
&\quad - \frac{(52p\ -\ w_1\ -\ 2w_2)^2}{9p^2}
\end{aligned}
$$

$$= \frac{4464p^2\ -\ 144pw_1\ -\ 72pw_2}{9p^2} \tag{52a}$$

$$+ \frac{2184p^2\ -\ 42pw_1\ -\ 84pw_2}{9p^2} \tag{52b}$$

$$- \frac{3844p^2\ -\ 248pw_1\ +\ 4w_1^2\ -\ 124pw_2\ +\ 4w_1w_2\ +\ w_2^2}{9p^2} \tag{52c}$$

$$+ \frac{3224p^2\ -\ 166pw_1\ +\ 2w_1^2\ -\ 176pw_2\ +\ 5w_1w_2\ +\ 2w_2^2}{9p^2} \tag{52d}$$

$$- \frac{2704p^2\ -\ 104pw1\ +\ w_1^2\ -\ 208pw_2\ +\ 4w_1w_2\ +\ 4w_2^2}{9p^2} \tag{52e}$$

Notice that we have terms in the following p^2, pw_1 pw_2, w_1^2, w_1w_2 and w_2^2. Now rearrange terms in equation 52 combining like terms.

$$
\begin{aligned}
y(p, w_1, w_2) \; &= \frac{4464p^2 \; + \; 2184p^2 - 3844p^2 \; + \; 3224p^2 \; - \; 2704p^2}{9p^2} && \text{(53a)}\\
&+ \frac{-144pw_1 \; - \; 42pw_1 \; + \; 248pw_1 \; - \; 166pw_1 \; + \; 104pw_1}{9p^2} && \text{(53b)}\\
&+ \frac{-72pw_2 \; - \; 84pw_2 \; + \; 124pw_2 \; - \; 176pw_2 \; + \; 208pw_2}{9p^2} && \text{(53c)}\\
&+ \; \frac{-4w_1^2 \; + \; 2w_1^2 \; - \; w_1^2}{9p^2} && \text{(53d)}\\
&+ \; \frac{-4w_1w_2 \; + \; 5w_1w_2 \; - \; 4w_1w_2}{9p^2} && \text{(53e)}\\
&+ \; \frac{-w_2^2 \; + \; 2w_2^2 \; - \; 4w_2^2}{9p^2} && \text{(53f)}
\end{aligned}
$$

Now combine terms

$$
\begin{aligned}
y(p, w_1, w_2) \; &= \frac{3324p^2}{9p^2} \; + \; \frac{0pw_1}{9p^2} \; + \; \frac{0pw_2}{9p^2} \; + \; \frac{-3w_1^2}{9p^2} \; + \; \frac{-3w_1w_2}{9p^2} \; + \; \frac{-3w_2^2}{9p^2}\\
&= \frac{1108p^2}{3p^2} \; - \; \frac{w_1^2}{3p^2} \; - \; \frac{w_1w_2}{3p^2} \; - \; \frac{w_2^2}{3p^2} && \text{(54)}\\
&= \frac{1108}{3} \; - \; \frac{w_1^2}{3p^2} \; - \; \frac{w_1w_2}{3p^2} \; - \; \frac{w_2^2}{3p^2}
\end{aligned}
$$

5.5. **The example numerical profit function.** We obtain the profit function by substituting the optimal values of x_1 and x_2 into the profit equation

$$
\begin{aligned}
\pi(p, w_1, w_2) \; &= pf(x_1(p, w_2, w_2), \; x_2(p, w_1, w_2)) \; - \; w_1x_1(p, w_2, w_2) \; - \; w_2x_2(p, w_2, w_2)\\
&= p\left(\frac{1108p^2}{3p^2} - \frac{w_1^2}{3p^2} - \frac{w_1w_2}{3p^2} - \frac{w_2^2}{3p^2}\right)\\
&- w_1\left(\frac{62p \; - \; 2w_1 \; - \; w_2}{3p}\right) \; - \; w_2\left(\frac{52p \; - \; w_1 \; - \; 2w_2}{3p}\right) && \text{(55)}\\
&= \frac{1108p^2 - w_1^2 - w_1w_2 - w_2^2 - 62pw_1 + 2w_1^2 \; + \; w_1w_2 - 52pw_2 + w_1w_2 + 2w_2^2}{3p}\\
&= \frac{1108p^2 + w_1^2 + w_2^2 + w_1w_2 - 62pw_1 - 52pw_2}{3p}
\end{aligned}
$$

5.6. **Optimal input demands via Hotelling's lemma.** We can find the optimal input demands by taking the derivative of equation 86 with respect to w_1 and w_2. First with respect to w_1

$$
\begin{aligned}
\pi(p, w_1, w_2) &= \frac{1108p^2 + w_1^2 + w_2^2 + w_1 w_2 - 62pw_1 - 52pw_2}{3p} \\
\frac{\partial \pi(p, w_1, w_2)}{\partial w_1} &= \frac{2w_1 + w_2 - 62p}{3p} \\
\Rightarrow\ x_1(p, w_1, w_2) &= -\frac{2w_1 + w_2 - 62p}{3p} = \frac{62p - 2w_1 - w_2}{3p}
\end{aligned} \tag{56}
$$

Then with respect to w_2

$$
\begin{aligned}
\frac{\partial \pi(p, w_1, w_2)}{\partial w_2} &= \frac{2w_2 + w_1 - 52p}{3p} \\
\Rightarrow\ x_2(p, w_1, w_2) &= -\frac{2w_2 + w_1 - 52p}{3p} = \frac{52p - w_1 - 2w_2}{3p}
\end{aligned} \tag{57}
$$

5.7. **Optimal output via Hotelling's lemma.** Take the derivative of the profit function with respect to p using the quotient rule

$$
\begin{aligned}
\pi(p, w_1, w_2) &= \frac{1108p^2 + w_1^2 + w_2^2 + w_1 w_2 - 62pw_1 - 52pw_2}{3p} \\
\frac{\partial \pi(p, w_1, w_2)}{\partial p} &= \frac{3p\,(2216p - 62w_1 - 52w_2)\ -\ 3\left(1108p^2 + w_1^2 + w_2^2 + w_1 w_2 - 62pw_1 - 52pw_2\right)}{9p^2} \\
&= \frac{6648p^2 - 186pw_1 - 156pw_2 - 3324p^2 - 3w_1^2 - 3w_2^2 - 3w_1 w_2 + 186pw_1 + 156pw_2}{9p^2} \\
&= \frac{3324p^2 - 3w_1^2 - 3w_2^2 - 3w_1 w_2}{9p^2} \\
\Rightarrow\ y(p, w_1, w_2) &= \frac{1108p^2 - w_1^2 - w_2^2 - w_1 w_2}{3p^2}
\end{aligned} \tag{58}
$$

6. Algebraic Example

6.1. **Production function.** Consider the following production function.

$$
\begin{aligned}
y &= f(x_1,\ x_2) \\
&= \alpha_1 x_1\ +\ \alpha_2 x_2\ +\ \beta_{11} x_1^2\ +\ \beta_{12} x_1\, x_2\ +\ \beta_{22} x_2^2
\end{aligned} \tag{59}
$$

The first and second partial derivatives are given by

$$\begin{aligned}
\frac{\partial f(x_1,\, x_2)}{\partial x_1} &= \alpha_1 \;+\; 2\beta_{11}x_1 \;+\; \beta_{12}x_2 \\
\frac{\partial f(x_1,\, x_2)}{\partial x_2} &= \alpha_2 \;+\; \beta_{12}x_1 \;+\; 2\beta_{22}x_2 \\
\frac{\partial^2 f(x_1,\, x_2)}{\partial x_1^2} &= 2\beta_{11} \\
\frac{\partial^2 f(x_1,\, x_2)}{\partial x_1 \partial x_2} &= \beta_{12} \\
\frac{\partial^2 f(x_1,\, x_2)}{\partial x_2^2} &= 2\beta_{22}
\end{aligned} \tag{60}$$

The Hessian is

$$\nabla^2 f(x_1,\, x_2) \;=\; \begin{pmatrix} 2\beta_{11} & \beta_{12} \\ \beta_{12} & 2\beta_{22} \end{pmatrix} \tag{61}$$

The determinant of the Hessian is given by

$$\begin{vmatrix} 2\beta_{11} & \beta_{12} \\ \beta_{12} & 2\beta_{11} \end{vmatrix} \;=\; 4\beta_{11}\beta_{22} \;-\; \beta_{12}^2 \tag{62}$$

The rate of technical substitution is given by

$$RTS \;=\; \frac{\partial x_2(y,\, x_1)}{\partial x_1} \;=\; -\frac{\alpha_1 \;+\; 2\beta_{11}x_1 \;+\; \beta_{12}x_2}{\alpha_2 \;+\; \beta_{12}x_1 \;+\; 2\beta_{22}x_2} \tag{63}$$

The elasticity substitution is given by

$$\begin{aligned}
\sigma_{12} &= \frac{-f_1\, f_2\, (x_1\, f_1 \;+\; x_2\, f_2)}{x_1\, x_2\, (f_{11}\, f_2^2 \;-\; 2f_{12}\, f_1\, f_2 \;+\; f_{22}\, f_1^2)} \\
&= \frac{-(\alpha_1 \;+\; 2x_1\beta_{11} \;+\; x_2\beta_{12})(\alpha_2 \;+\; x_1\beta_{12} \;+ 2x_2\beta_{22})\,(2x_1^2\beta_{11} \;+\; x_1(\alpha_1 \;+\; 2x_2\beta_{12}) \;+\; x_2(\alpha_2 \;+\; 2x_2\beta_{22}))}{x_1\, x_2\; (2\,(\alpha_1 + 2x_1\beta_{11} + x_2\beta_{12})^2\, \beta_{22} - 2\beta_{12}\,(\alpha_1 + 2x_1\beta_{11} + x_2\beta_{12})\,(\alpha_2 + x_1\beta_{12} + 2x_2\beta_{22}) + 2\beta_{11}(\alpha_2 + x_1\beta_{12} + 2x_2\beta_{22})^2))}
\end{aligned} \tag{64}$$

6.2. **Profit maximization.** Profit is given by

$$\begin{aligned}
\pi \;&= p\, f(x_1,\, x_2) \;-\; w_1\, x_1 \;-\; w_2\, x_2 \\
&= p\, \left[\, \alpha_1 x_1 \;+\; \alpha_2 x_2 \;+\; \beta_{11}x_1^2 \;+\; \beta_{12}x_1\, x_2 \;+\; \beta_{22}x_2^2 \,\right] \;-\; w_1\, x_1 \;-\; w_2\, x_2
\end{aligned} \tag{65}$$

We maximize profit by taking the derivatives of 65 setting them equal to zero and solving for x_1 and x_2.

$$\begin{aligned}
\frac{\partial \pi}{\partial x_1} &= p\,[\, \alpha_1 \;+\; 2\beta_{11}x_1 \;+\; \beta_{12}x_2 \,] \;-\; w_1 \;=\; 0 \\
\frac{\partial \pi}{\partial x_2} &= p\,[\, \alpha_2 \;+\; \beta_{12}x_1 \;+\; 2\beta_{22}x_2 \,] \;-\; w_2 \;=\; 0
\end{aligned} \tag{66}$$

Rearranging 66 we obtain

$$\alpha_1 \;+\; 2\beta_{11}x_1 \;+\; \beta_{12}x_2 \;=\; \frac{w_1}{p} \tag{67a}$$

$$\alpha_2 \;+\; \beta_{12}x_1 \;+\; 2\beta_{22}x_2 \;=\; \frac{w_2}{p} \tag{67b}$$

Now solve equation 67a for x_1 as follows

$$x_1 \;=\; \frac{w_1 - p\alpha_1 - px_2\beta_{12}}{2p\beta_{11}} \tag{68}$$

Then substitute x_1 from equation 68 into equation 67b as follows

$$\begin{aligned}
\alpha_2 \;+\; \beta_{12}\left(\frac{w_1 - p\alpha_1 - px_2\beta_{12}}{2p\beta_{11}}\right) \;+\; 2\beta_{22}x_2 \;&=\; \frac{w_2}{p} \\
\Rightarrow \frac{2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12} \;-\; px_2\beta_{12}^2 \;+\; 4px_2\beta_{11}\beta_{22}}{2p\beta_{11}} \;&=\; \frac{w_2}{p} \\
\Rightarrow \frac{2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12}}{2p\beta_{11}} \;+\; \frac{4px_2\beta_{11}\beta_{22} \;-\; px_2\beta_{12}^2}{2p\beta_{11}} \;&=\; \frac{w_2}{p} \\
\Rightarrow \frac{2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12}}{2p\beta_{11}} \;+\; \frac{x_2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}{2\beta_{11}} \;&=\; \frac{w_2}{p}
\end{aligned} \tag{69}$$

Now isolate x_2 on the left hand side of equation 69 as follows.

$$\begin{aligned}
\frac{x_2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}{2\beta_{11}} \;&=\; \frac{w_2}{p} \;-\; \frac{2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12}}{2p\beta_{11}} \\
\Rightarrow x_2 \;&=\; \frac{2\beta_{11}w_2}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \;-\; \frac{(2\beta_{11})\left(2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12}\right)}{2p\beta_{11}\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{2\beta_{11}w_2}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \;-\; \frac{\left(2p\alpha_2\beta_{11} \;+\; w_1\beta_{12} \;-\; p\alpha_1\beta_{12}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{2\beta_{11}w_2 \;-\; 2p\alpha_2\beta_{11} \;-\; w_1\beta_{12} \;+\; p\alpha_1\beta_{12}}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{2\beta_{11}w_2 \;-\; w_1\beta_{12} \;+\; p\left(\alpha_1\beta_{12} \;-\; 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}
\end{aligned} \tag{70}$$

If we substitute the last expression in equation 70 for x_2 in equation 68 we obtain

$$\begin{aligned}
x_1 \;&= \frac{w_1 - p\alpha_1 - px_2\beta_{12}}{2p\beta_{11}} \\
&= \frac{w_1 - p\alpha_1}{2p\beta_{11}} \;-\; \frac{p\beta_{12}}{2p\beta_{11}}\,x_2 \\
&= \frac{w_1 - p\alpha_1}{2p\beta_{11}} \;-\; \frac{p\beta_{12}}{2p\beta_{11}}\left(\frac{2\beta_{11}w_2 \;-\; 2p\alpha_2\beta_{11} \;-\; w_1\beta_{12} \;+\; p\alpha_1\beta_{12}}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right)
\end{aligned} \tag{71}$$

Now put both terms in equation 71 over a common denominator and simplify

$$
\begin{aligned}
x_1 &= \frac{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)(w_1 - p\alpha_1) \;-\; p\beta_{12}\left(2\beta_{11}w_2 \;-\; 2p\alpha_2\beta_{11} \;-\; w_1\beta_{12} \;+\; p\alpha_1\beta_{12}\right)}{2p\beta_{11}p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)(w_1 - p\alpha_1) \;-\; \beta_{12}\left(2\beta_{11}w_2 - 2p\alpha_2\beta_{11} - w_1\beta_{12} + p\alpha_1\beta_{12}\right)}{2p\beta_{11}\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{4w_1\beta_{11}\beta_{22} - w_1\beta_{12}^2 - 4p\alpha_1\beta_{11}\beta_{22} + p\alpha_1\beta_{12}^2 - 2\beta_{11}\beta_{12}w_2 + 2p\alpha_2\beta_{11}\beta_{12} + w_1\beta_{12}^2 - p\alpha_1\beta_{12}^2}{2p\beta_{11}\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{4w_1\beta_{11}\beta_{22} - 4p\alpha_1\beta_{11}\beta_{22} - 2\beta_{11}\beta_{12}w_2 \;+\; 2p\alpha_2\beta_{11}\beta_{12}}{2p\beta_{11}\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \;+\; p\alpha_2\beta_{12}}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)} \\
&= \frac{2w_1\beta_{22} - \beta_{12}w_2 \;+\; p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}
\end{aligned}
\tag{72}
$$

6.3. **Necessary and sufficient conditions for a maximum.** Consider the Hessian of the profit equation.

$$
\nabla^2\,\pi(x_1,\,x_2) \;=\; p\begin{pmatrix} 2\beta_{11} & \beta_{12} \\ \beta_{12} & 2\beta_{22} \end{pmatrix} \;=\; \begin{pmatrix} 2p\beta_{11} & p\beta_{12} \\ p\beta_{12} & 2p\beta_{22} \end{pmatrix} \tag{73}
$$

For a maximum we need the diagonal elements to be negative and the determinant to be positive. The diagonal elements will negative if β_{11} and β_{22} are negative. The determinant of the Hessian is given by

$$
\begin{vmatrix} 2p\beta_{11} & p\beta_{12} \\ p\beta_{12} & 2p\beta_{22} \end{vmatrix} \;=\; 4p^2\beta_{11}\beta_{22} \;-\; p^2\beta_{12}^2 \;=\; p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right) \tag{74}
$$

So the solution will be a maximum if β_{11} and β_{22} are both less than zero and $4\beta_{11}\beta_{22} \;>\; \beta_{12}^2$. Notice that if $4\beta_{11}\beta_{22} \;=\; \beta_{12}^2$, the test fails.

6.4. **Optimal output.** We can obtain optimal output as a function of p, w_1 and w_2 by substituting the optimal values of x_1 and x_2 into the production function.

$$
\begin{aligned}
y(p, w_1, w_2) &= f(x_1(p, w_2, w_2),\ x_2(p, w_1, w_2)) \\
&= \alpha_1 \left(\frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \ + \ p\alpha_2\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right) \\
&+ \alpha_2 \left(\frac{2\beta_{11}w_2 \ - \ 2p\alpha_2\beta_{11} \ - \ w_1\beta_{12} \ + \ p\alpha_1\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right) \\
&+ \beta_{11} \left(\frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \ + \ p\alpha_2\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right)^2 \\
&+ \beta_{12} \left(\left(\frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \ + \ p\alpha_2\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right) \left(\frac{2\beta_{11}w_2 \ - \ 2p\alpha_2\beta_{11} \ - \ w_1\beta_{12} \ + \ p\alpha_1\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right) \right) \\
&\quad + \ \beta_{22} \left(\frac{2\beta_{11}w_2 \ - \ 2p\alpha_2\beta_{11} \ - \ w_1\beta_{12} \ + \ p\alpha_1\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right)^2
\end{aligned}
\tag{75}
$$

We can simplify equation 75 by simplifying the squared and cross product terms. First consider x_1^2.

$$
\begin{aligned}
x_1^2 \ &= \ \left(\frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \ + \ p\alpha_2\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12})^2} \right) \left(\frac{2w_1\beta_{22} - 2p\alpha_1\beta_{22} - \beta_{12}w_2 \ + \ p\alpha_2\beta_{12}}{p\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)} \right) \\
&= \ \left(\frac{4w_1^2\beta_{22}^2 - 4pw_1\alpha_1\beta_{22}^2 - 2w_1w_2\beta_{12}\beta_{22} \ + \ 2pw_1\alpha_2\beta_{12}\beta_{22}}{p^2\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)^2} \right) \\
&+ \ \left(\frac{-4pw_1\alpha_1\beta_{22}^2 \ + \ 4p^2\alpha_1^2\beta_{22}^2 + 2pw_2\alpha_1\beta_{12}\beta_{22} \ - \ 2p^2\alpha_1\alpha_2\beta_{12}\beta_{22}}{p^2\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)^2} \right) \\
&+ \ \left(\frac{-2w_1w_2\beta_{12}\beta_{22} \ + \ 2p\alpha_1w_2\beta_{12}\beta_{22} + w_2^2\beta_{12}^2 \ - \ pw_2\alpha_2\beta_{12}^2}{p^2\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)^2} \right) \\
&+ \ \left(\frac{2pw_1\alpha_2\beta_{12}\beta_{22} - 2p^2\alpha_1\alpha_2\beta_{12}\beta_{22} - pw_2\alpha_2\beta_{12}^2 \ + \ p^2\alpha_2^2\beta_{12}^2}{p^2\,(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2)^2} \right)
\end{aligned}
\tag{76}
$$

We can simplify equation 76 as follows

$$
\begin{aligned}
x_1^2 \;=\; & \left(\frac{4w_1^2\beta_{22}^2 - 8pw_1\alpha_1\beta_{22}^2 - 4w_1w_2\beta_{12}\beta_{22} \;+\; 4pw_1\alpha_2\beta_{12}\beta_{22}}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{4p^2\alpha_1^2\beta_{22}^2 + 4pw_2\alpha_1\beta_{12}\beta_{22} \;-\; 4p^2\alpha_1\alpha_2\beta_{12}\beta_{22}}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{w_2^2\beta_{12}^2 - 2pw_2\alpha_2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{p^2\alpha_2^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right)
\end{aligned}
\tag{77}
$$

Simplifying again we obtain

$$
\begin{aligned}
x_1^2 \;=\; & \left(\frac{4p^2\alpha_1^2\beta_{22}^2 \;-\; 4p^2\alpha_1\alpha_2\beta_{12}\beta_{22} \;+\; p^2\alpha_2^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{4pw_1\alpha_2\beta_{12}\beta_{22} - 8pw_1\alpha_1\beta_{22}^2 + 4pw_2\alpha_1\beta_{12}\beta_{22} - 2p\alpha_2w_2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{4w_1^2\beta_{22}^2 - 4w_1w_2\beta_{12}\beta_{22} \;+\; w_2^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
=\; & \left(\frac{p^2\left(4\alpha_1^2\beta_{22}^2 \;-\; 4\alpha_1\alpha_2\beta_{12}\beta_{22} \;+\; \alpha_2^2\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{4pw_1\left(\alpha_2\beta_{12}\beta_{22} - 2\alpha_1\beta_{22}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{2pw_2\left(2\alpha_1\beta_{12}\beta_{22} - \alpha_2\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right) \\
& + \left(\frac{4w_1^2\beta_{22}^2 - 4w_1w_2\beta_{12}\beta_{22} \;+\; w_2^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)^2}\right)
\end{aligned}
\tag{78}
$$

Similarly for x_2

$$
\begin{aligned}
x_2^2 \ = \ & \left(\frac{4w_2^2\beta_{11}^2 - 8pw_2\alpha_2\beta_{11}^2 + 4p^2\alpha_2^2\beta_{11}^2 \ - \ 4w_1w_2\beta_{11}\beta_{12}}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{4p\alpha_1w_2\beta_{11}\beta_{12} + 4pw_1\alpha_2\beta_{11}\beta_{12} \ - \ 4p^2\alpha_1\alpha_2\beta_{11}\beta_{12}}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{w_1^2\beta_{12}^2 - 2pw_1\alpha_1\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{p^2\alpha_1^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
= \ & \left(\frac{p^2\left(4\alpha_2^2\beta_{11}^2 \ - \ 4\alpha_1\alpha_2\beta_{11}\beta_{12} \ + \ \alpha_1^2\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{2pw_1\left(2\alpha_2\beta_{11}\beta_{12} - \alpha_1\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{4pw_2\left(2\alpha_1\beta_{11}\beta_{12} - 2\alpha_2\beta_{11}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{11}^2\right)^2} \right) \\
+ \ & \left(\frac{w_1^2\beta_{12}^2 - 4w_1w_2\beta_{11}\beta_{12} \ + \ 4w_2^2\beta_{11}^2}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right)
\end{aligned} \tag{79}
$$

And for $x_1 * x_2$

$$
\begin{aligned}
x_1 * x_2 \ = \ & \left(\frac{2w_1\beta_{22} - \beta_{12}w_2 \ + \ p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)} \right) \left(\frac{2\beta_{11}w_2 \ - \ w_1\beta_{12} \ + \ p\left(\alpha_1\beta_{12} \ - \ 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)} \right) \\
= \ & \left(\frac{4w_1w_2\beta_{11}\beta_{22} - 2w_1^2\beta_{12}\beta_{22} + 2pw_1\beta_{22}\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{-2w_2^2\beta_{11}\beta_{12} + w_1w_2\beta_{12}^2 - pw_2\beta_{12}\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
+ \ & \left(\frac{2pw_2\beta_{11}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - pw_1\beta_{12}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) + p^2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)} \right)
\end{aligned} \tag{80}
$$

Now collect terms in equation 80

$$
\begin{aligned}
x_1 * x_2 \ =& \left(\frac{p^2\left(\alpha_1\beta_{12}-2\alpha_2\beta_{11}\right)\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)+2pw_1\beta_{22}\left(\alpha_1\beta_{12}-2\alpha_2\beta_{11}\right)-pw_2\beta_{12}\left(\alpha_1\beta_{12}-2\alpha_2\beta_{11}\right)}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
&+\left(\frac{2pw_2\beta_{11}\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)-pw_1\beta_{12}\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
&+\left(\frac{-2w_1^2\beta_{12}\beta_{22}+w_1w_2\beta_{12}^2+4w_1w_2\beta_{11}\beta_{22}-2w_2^2\beta_{11}\beta_{12}}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
=& \left(\frac{p^2\left(\alpha_1\beta_{12}-2\alpha_2\beta_{11}\right)\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
&+\left(\frac{pw_1\left(2\beta_{22}\left(\alpha_1\beta_{12}-2\alpha_2\beta_{11}\right)-\beta_{12}\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)\right)}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
&+\left(\frac{pw_2\left(2\beta_{11}\left(\alpha_2\beta_{12}-2\alpha_1\beta_{22}\right)-\beta_{12}\left(\alpha_1\beta_{12}-2\alpha_1\beta_{11}\right)\right)}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right) \\
&+\left(\frac{-2w_1^2\beta_{12}\beta_{22}+w_1w_2\left(4\beta_{11}\beta_{22}+\beta_{12}^2\right)-2w_2^2\beta_{11}\beta_{12}}{p^2\left(4\beta_{11}\beta_{22}\ -\ \beta_{12}^2\right)^2}\right)
\end{aligned} \tag{81}
$$

Now substitute the results from equations 70, 72 78 , 79 and 81 into equation 75

$$
\begin{aligned}
y(p,w_1,w_2) &= f(x_1(p,w_2,w_2),\ x_2(p,w_1,w_2)) \\
&= \alpha_1 \left(\frac{2w_1\beta_{22} - \beta_{12}w_2 \ + \ p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)} \right) \\
&+ \alpha_2 \left(\frac{2\beta_{11}w_2 \ - \ w_1\beta_{12} \ + \ p\left(\alpha_1\beta_{12} \ - \ 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)} \right) \\
&+\beta_{11} \left(\frac{p^2\left(4\alpha_1^2\beta_{22}^2 - 4\alpha_1\alpha_2\beta_{12}\beta_{22} + \alpha_2^2\beta_{12}^2\right) + 4pw_1\left(\alpha_2\beta_{12}\beta_{22} - 2\alpha_1\beta_{22}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+\beta_{11} \left(\frac{2pw_2\left(2\alpha_1\beta_{12}\beta_{22} - \alpha_2\beta_{12}^2\right) + 4w_1^2\beta_{22}^2 - 4w_1w_2\beta_{12}\beta_{22} + w_2^2\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \beta_{12} \left(\frac{p^2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} + \frac{pw_1\left(2\beta_{22}\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - \beta_{12}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
&+ \beta_{12} \left(\frac{pw_2\left(2\beta_{11}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - \beta_{12}\left(\alpha_1\beta_{12} - 2\alpha_1\beta_{11}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} + \frac{-2w_1^2\beta_{12}\beta_{22} + w_1w_2\left(4\beta_{11}\beta_{22} + \beta_{12}^2\right) - 2w_2^2\beta_{11}\beta_{12}}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
&+ \ \beta_{22} \left(\frac{p^2\left(4\alpha_2^2\beta_{11}^2 \ - \ 4\alpha_1\alpha_2\beta_{11}\beta_{12} \ + \ \alpha_1^2\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} + \frac{2pw_1\left(2\alpha_2\beta_{11}\beta_{12} - \alpha_1\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right) \\
&+ \ \beta_{22} \left(\frac{4pw_2\left(2\alpha_1\beta_{11}\beta_{12} - 2\alpha_2\beta_{11}^2\right)}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{11}^2\right)^2} + \frac{w_1^2\beta_{12}^2 - 4w_1w_2\beta_{11}\beta_{12} \ + \ 4w_2^2\beta_{11}^2}{p^2\left(4\beta_{11}\beta_{22} \ - \ \beta_{12}^2\right)^2} \right)
\end{aligned}
\tag{82}
$$

To start simplifying, we need to put everything over a common denominator

$$
\begin{aligned}
y(p, w_1, w_2) &= f(x_1(p, w_2, w_2),\ x_2(p, w_1, w_2)) \\
&= \left(\frac{2pw_1\alpha_1\beta_{22}\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right) - pw_2\alpha_1\beta_{12}\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right) + p^2\alpha_1\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{2pw_2\alpha_2\beta_{11}\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right) - pw_1\alpha_2\beta_{12}\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right) + p^2\alpha_2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{p^2\beta_{11}\left(4\alpha_1^2\beta_{22}^2 - 4\alpha_1\alpha_2\beta_{12}\beta_{22} + \alpha_2^2\beta_{12}^2\right) + 4pw_1\beta_{11}\left(\alpha_2\beta_{12}\beta_{22} - 2\alpha_1\beta_{22}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{2pw_2\beta_{11}\left(2\alpha_1\beta_{12}\beta_{22} - \alpha_2\beta_{12}^2\right) + 4w_1^2\beta_{11}\beta_{22}^2 - 4w_1w_2\beta_{11}\beta_{12}\beta_{22} + w_2^2\beta_{11}\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{p^2\beta_{12}\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} + \frac{pw_1\beta_{12}\left(2\beta_{22}\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - \beta_{12}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{pw_2\,\beta_{12}\left(2\beta_{11}\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - \beta_{12}\left(\alpha_1\beta_{12} - 2\alpha_1\beta_{11}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{-2w_1^2\beta_{12}^2\beta_{22} + w_1w_2\beta_{12}\left(4\beta_{11}\beta_{22} + \beta_{12}^2\right) - 2w_2^2\beta_{11}\beta_{12}^2}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{p^2\beta_{22}\left(4\alpha_2^2\beta_{11}^2 - 4\alpha_1\alpha_2\beta_{11}\beta_{12} + \alpha_1^2\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} + \frac{2pw_1\beta_{22}\left(2\alpha_2\beta_{11}\beta_{12} - \alpha_1\beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right) \\
&+ \left(\frac{4pw_2\beta_{22}\left(2\alpha_1\beta_{11}\beta_{12} - 2\alpha_2\beta_{11}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{11}^2\right)^2} + \frac{w_1^2\beta_{12}^2\beta_{22} - 4w_1w_2\beta_{11}\beta_{12}\beta_{22} + 4w_2^2\beta_{11}^2\beta_{22}}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \right)
\end{aligned} \tag{83}
$$

Notice that we have terms in the following p^2, pw_1 pw_2, w_1^2, w_1w_2 and w_2^2. Now rearrange terms in equation 83 combining like terms. After some manipulations we obtain

$$
y(p, w_1, w_2) = \frac{w_2^2\beta_{11} - w_1w_2\beta_{12} + w_1^2\beta_{22} - p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \tag{84}
$$

6.5. **The example profit function.** We obtain the profit function by substituting the optimal values of x_1 and x_2 into the profit equation

$$
\begin{aligned}
\pi(p, w_1, w_2) &= pf(x_1(p, w_2, w_2),\ x_2(p, w_1, w_2)) - w_1x_1(p, w_2, w_2) - w_2x_2(p, w_2, w_2) \\
&= p\left(\frac{w_2^2\beta_{11} - w_1w_2\beta_{12} + w_1^2\beta_{22} - p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \right) \\
&- w_1\left(\frac{2w_1\beta_{22} - \beta_{12}w_2 + p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \right) - w_2\left(\frac{2\beta_{11}w_2 - w_1\beta_{12} + p\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \right)
\end{aligned} \tag{85}
$$

We can simplify equation 85 as follows

$$\begin{aligned}
\pi(p, w_1, w_2) &= \left(\frac{w_2^2\beta_{11} - w_1 w_2 \beta_{12} + w_1^2 \beta_{22} \;-\; p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&\quad - w_1\left(\frac{2w_1\beta_{22} - \beta_{12}w_2 + p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \;-\; w_2\left(\frac{2\beta_{11}w_2 - w_1\beta_{12} + p\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&= \left(\frac{w_2^2\beta_{11} - w_1 w_2 \beta_{12} + w_1^2 \beta_{22} \;-\; p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&\quad + \left(\frac{-2w_1^2\beta_{22} + w_1 w_2\beta_{12} - p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&\quad + \left(\frac{-2w_2^2\beta_{11} + w_1 w_2\beta_{12} - p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&= \left(\frac{-w_2^2\beta_{11} + w_1 w_2 \beta_{12} - w_1^2 \beta_{22} \;-\; p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&\quad + \left(\frac{-p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right) \\
&\quad + \left(\frac{-p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} \;-\; \beta_{12}^2\right)}\right)
\end{aligned} \tag{86}$$

We can write equation 86 in the following useful fashion

$$\pi(p, w_1, w_2) = \left(\frac{-w_2^2\beta_{11} + w_1 w_2\beta_{12} - w_1^2\beta_{22} - p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}\right) \tag{87}$$

6.6. **Optimal input demands.** We can find the optimal input demands by taking the derivative of equation 86 with respect to w_1 and w_2. First with respect to w_1

$$\begin{aligned}
\frac{\partial \pi(p, w_1, w_2)}{\partial w_1} &= \left(\frac{\beta_{12}w_2 - 2\beta_{22}w_1 - p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}\right) \\
\Rightarrow\; x_1 &= \left(\frac{-\beta_{12}w_2 + 2\beta_{22}w_1 + p\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}\right)
\end{aligned} \tag{88}$$

Then with respect to w_2

$$\begin{aligned}
\frac{\partial \pi(p, w_1, w_2)}{\partial w_2} &= \left(\frac{-2w_2\beta_{11} + w_1\beta_{12} - p\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}\right) \\
\Rightarrow\; x_2 &= \left(\frac{2w_2\beta_{11} - w_1\beta_{12} + p\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}\right)
\end{aligned} \tag{89}$$

6.7. **Optimal output.** Take the derivative of equation 87 with respect to p

$$\pi(p, w_1, w_2) = \left(\frac{-w_2^2\beta_{11} + w_1 w_2 \beta_{12} - w_1^2 \beta_{22} - p w_1 \left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - p w_2 \left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - p^2 \left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \right)$$

$$\begin{aligned}
\frac{\partial \pi(p, w_1, w_2)}{\partial p} &= \frac{\left(p\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)\right)\left(-w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - 2p\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \\
&- \frac{\left(-w_2^2\beta_{11} + w_1 w_2\beta_{12} - w_1^2\beta_{22} - p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)\right)\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \\
&= \frac{\left(-p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) - p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) - 2p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)\right)\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \\
&+ \frac{\left(w_2^2\beta_{11} - w_1 w_2\beta_{12} + w_1^2\beta_{22} + p w_1\left(\alpha_2\beta_{12} - 2\alpha_1\beta_{22}\right) + p w_2\left(\alpha_1\beta_{12} - 2\alpha_2\beta_{11}\right) + p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)\right)\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \\
&= \frac{\left(-p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right) + \left(w_2^2\beta_{11} - w_1 w_2\beta_{12} + w_1^2\beta_{22}\right)\right)\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)^2} \\
&= \frac{\left(-p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right) + \left(w_2^2\beta_{11} - w_1 w_2\beta_{12} + w_1^2\beta_{22}\right)\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)} \\
&= \frac{w_2^2\beta_{11} - w_1 w_2\beta_{12} + w_1^2\beta_{22} - p^2\left(\alpha_2^2\beta_{11} - \alpha_1\alpha_2\beta_{12} + \alpha_1^2\beta_{22}\right)}{p^2\left(4\beta_{11}\beta_{22} - \beta_{12}^2\right)}
\end{aligned} \tag{90}$$

7. SINGULARITY OF THE HESSIAN MATRIX

We can show that the Hessian matrix of the profit function is singular. This is a direct implication of linear homogeneity. To show this first write the identity

$$\pi(t\,p,\, tw\,) \;=\; t\,\pi(p,\, w), \quad \text{where t is a scaler greater than zero} \tag{91}$$

Differentiate equation 91 with respect to t. This will yield

$$\sum_{j=1}^{m} \frac{\partial \pi}{\partial (t\,p_j)}\, p_j \;+\; \sum_{i=1}^{n} \frac{\partial \pi}{\partial (t\,w_i)}\, w_i \;=\; \pi(p,\, w) \tag{92}$$

Differentiate equation 92 with respect to t again. Each term in equation 92 will yield a sum of derivatives so the result will be a double sum as follows

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=1}^{m} \frac{\partial^2 \pi}{\partial (t\,p_k)\,\partial (t\,p_j)}\, p_k\, p_j \;+\; \sum_{j=1}^{m}\sum_{\ell=1}^{n} \frac{\partial^2 \pi}{\partial (t\,w_\ell)\,\partial (t\,p_j)}\, w_\ell\, p_j \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{\partial^2 \pi}{\partial (t\,p_k)\,\partial (t\,w_i)}\, p_k\, w_i \;+\; \sum_{i=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 \pi}{\partial (t\,w_\ell)\,\partial (t\,w_i)}\, w_\ell\, w_i \;=\; 0
\end{aligned} \tag{93}$$

We can factor t out of each terms in equation 93 because

$$\frac{\partial\, f(x)}{\partial\,(t\,x)} \;=\; \frac{1}{t}\,\frac{\partial\, f(x)}{\partial\, x} \tag{94}$$

This will give

$$\begin{aligned}
&\frac{1}{t^2}\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{\partial^2\pi}{\partial p_k\,\partial p_j}\,p_k\,p_j \;+\; \frac{1}{t^2}\sum_{j=1}^{m}\sum_{\ell=1}^{n}\frac{\partial^2\pi}{\partial w_\ell\,\partial p_j}\,w_\ell\,p_j \\
&\quad + \frac{1}{t^2}\sum_{i=1}^{n}\sum_{k=1}^{m}\frac{\partial^2\pi}{\partial p_k\,\partial w_i}\,p_k\,w_i \;+\; \frac{1}{t^2}\sum_{i=1}^{n}\sum_{\ell=1}^{n}\frac{\partial^2\pi}{\partial w_\ell\,\partial w_i}\,w_\ell\,w_i \;=\; 0
\end{aligned} \tag{95}$$

This then implies that

$$\begin{aligned}
&\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{\partial^2\pi}{\partial p_k\,\partial p_j}\,p_k\,p_j \;+\; \sum_{j=1}^{m}\sum_{\ell=1}^{n}\frac{\partial^2\pi}{\partial w_\ell\,\partial p_j}\,w_\ell\,p_j \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{m}\frac{\partial^2\pi}{\partial p_k\,\partial w_i}\,p_k\,w_i \;+\; \sum_{i=1}^{n}\sum_{\ell=1}^{n}\frac{\partial^2\pi}{\partial w_\ell\,\partial w_i}\,w_\ell\,w_i \;=\; 0
\end{aligned} \tag{96}$$

The last expression in equation 96 is actually a quadratic form involving the Hessian of π(p,w). To see this write out the Hessian and then pre and post multiply by a vector containing output and input prices.

$$\begin{bmatrix} p_1 & p_2 & \dots & p_m & w_1 & \dots & w_n \end{bmatrix}
\begin{bmatrix}
\frac{\partial^2\pi}{\partial p_1^2} & \frac{\partial^2\pi}{\partial p_2\partial p_1} & \cdots & \frac{\partial^2\pi}{\partial p_m\partial p_1} & \frac{\partial^2\pi}{\partial w_1\partial p_1} & \cdots & \frac{\partial^2\pi}{\partial w_n\partial p_1} \\
\frac{\partial^2\pi}{\partial p_1\partial p_2} & \frac{\partial^2\pi}{\partial p_2^2} & \cdots & \frac{\partial^2\pi}{\partial p_m\partial p_2} & \frac{\partial^2\pi}{\partial w_1\partial p_2} & \cdots & \frac{\partial^2\pi}{\partial w_n\partial p_2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\
\frac{\partial^2\pi}{\partial p_1\partial p_m} & \frac{\partial^2\pi}{\partial p_2\partial p_m} & \cdots & \frac{\partial^2\pi}{\partial p_m^2} & \frac{\partial^2\pi}{\partial w_1\partial p_m} & \cdots & \frac{\partial^2\pi}{\partial w_n\partial p_m} \\
\frac{\partial^2\pi}{\partial p_1\partial w_1} & \frac{\partial^2\pi}{\partial p_2\partial w_1} & \cdots & \frac{\partial^2\pi}{\partial p_m\partial w_1} & \frac{\partial^2\pi}{\partial w_1^2} & \cdots & \frac{\partial^2\pi}{\partial w_n\partial w_1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \\
\frac{\partial^2\pi}{\partial p_1\partial w_n} & \frac{\partial^2\pi}{\partial p_2\partial w_n} & \cdots & \frac{\partial^2\pi}{\partial p_m\partial w_n} & \frac{\partial^2\pi}{\partial w_1\partial w_n} & \cdots & \frac{\partial^2\pi}{\partial w_n^2}
\end{bmatrix}
\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \tag{97}$$

Equation 96 implies that this quadratic form is zero for this particular vector (p,w). This means that the Hessian matrix is singular. The profit function is convex so we know that its Hessian is positive semi-definite. This implies that it is positive semidefinite only and cannot be positive definite.

8. Sensitivity Analysis for the Profit Maximization Problem

The Hessian of the profit function is positive semidefinite so all its diagonal elements are non-negative (Hadley [9, p. 117]).

8.1. **Response of Output Supply.**

Consider first the response of any output to its own price.

$$\frac{\partial y_j}{\partial p_j} \;=\; \frac{\partial^2\pi}{\partial p_j^2} \;\geq\; 0 \tag{98}$$

Supply curves derived from profit maximization will always slope upwards because this derivative is on the diagonal of the Hessian. Now consider the response of an output to another product price.

$$\frac{\partial y_j}{\partial p_k} \;=\; \frac{\partial^2\pi}{\partial p_k\,\partial p_j} \tag{99}$$

This element is not on the diagonal of the Hessian of the profit function and so we cannot determine its sign.

Similarly we cannot sign the response of an output to an input price.

$$\frac{\partial y_j}{\partial w_i} = \frac{\partial^2 \pi}{\partial w_i \, \partial p_j} \tag{100}$$

8.2. **Response of Input Supply.** Consider first the response of any input to its own price.

$$\frac{\partial x_i}{\partial w_i} = -\frac{\partial^2 \pi}{\partial w_i^2} \leq 0 \tag{101}$$

Demand curves derived from profit maximization will always slope downwards because this derivative is on the diagonal of the Hessian and input demand is the negative of the derivative of the profit function with respect to input price. Now consider the response of an input to another input price.

$$\frac{\partial x_i}{\partial w_\ell} = -\frac{\partial^2 \pi}{\partial w_\ell \, \partial w_i} \tag{102}$$

This element is not on the diagonal of the Hessian of the profit function and so we cannot determine its sign.

Similarly we cannot sign the response of an input to an output price.

$$\frac{\partial x_i}{\partial p_k} = -\frac{\partial^2 \pi}{\partial p_k \, \partial w_i} \tag{103}$$

8.3. **Homogeneity.** Because output supply and input demand are derivatives of a function which is homogeneous of degree one, they are homogeneous of degree zero.

$$y_j(\lambda p, \lambda w) = y_j(p, w) x_i(\lambda p, \lambda w) = x_i(p, w) \tag{104}$$

8.4. **Symmetry of response.** If π(p, w) is twice differentiable then Young's theorem on second cross partial derivatives implies

$$\begin{aligned} \frac{\partial^2 \pi}{\partial p_i \partial p_j} &= \frac{\partial^2 \pi}{\partial p_j \partial p_i} \\ \frac{\partial^2 \pi}{\partial p_k \partial w_i} &= \frac{\partial^2 \pi}{\partial w_i \partial p_k} \\ \frac{\partial^2 \pi}{\partial w_i \partial w_j} &= \frac{\partial^2 \pi}{\partial w_j \partial w_i} \end{aligned} \tag{105}$$

This then implies

$$\begin{aligned} \frac{\partial y_j}{\partial p_i} &= \frac{\partial y_i}{\partial p_j} \\ -\frac{\partial w_i}{\partial p_k} &= \frac{\partial y_k}{\partial w_i} \\ \frac{\partial x_i}{\partial w_j} &= \frac{\partial x_j}{\partial w_i} \end{aligned} \tag{106}$$

The response of any output to a different output price is symmetric, i.e. cross price derivatives are equal. Similarly with input demand response. The way an output responds to a particular input price is the same as the response of that input to the output's price.

8.5. **Own and cross price response.** Because any principal submatrix of a positive semi-definite matrix is also positive semidefinite (Hadley [9, p. 262]), we also have the following results.

$$\frac{\partial x_i}{\partial w_i} \cdot \frac{\partial x_j}{\partial w_j} \geq \frac{\partial x_i}{\partial w_j} \cdot \frac{\partial x_j}{\partial w_i} \tag{107}$$

To see this note that if the submatrix above is positive semi-definite then the determinant must be non-negative which means that (the product of the diagonal elements) - (the product of the off diagonal elements) must be greater than or equal to zero.

We also have

$$\frac{\partial y_j}{\partial p_j} \cdot \frac{\partial y_k}{\partial p_k} \geq \frac{\partial y_j}{\partial p_k} \cdot \frac{\partial y_k}{\partial p_j} \tag{108}$$

The product of own price responses is larger than the product in cross price responses.

We also have

$$-\frac{\partial y_j}{\partial p_j} \cdot \frac{\partial x_k}{\partial w_k} \geq -\frac{\partial y_j}{\partial w_k} \cdot \frac{\partial x_k}{\partial p_j} \tag{109}$$

9. Substitutes and Complements

9.1. **Gross substitutes.** We say that inputs are gross substitutes if

$$\frac{\partial x_i\,(p,\ w)}{\partial w_j} = \frac{\partial x_j\,(p,\ w)}{\partial w_i} \geq 0 \tag{110}$$

9.2. **Gross complements.** We say that inputs are gross complements if

$$\frac{\partial x_i\,(p,\ w)}{\partial w_j} = \frac{\partial x_j\,(p,\ w)}{\partial w_i} \leq 0 \tag{111}$$

10. Recovering Production Relationships from the Profit Function

10.1. **Marginal Products.** We can find marginal products from the first order conditions for profit maximization. The easiest way to see this is when technology is represented by an asymmetric transformation function.

$$\pi = \max_{x\ \tilde{y}}\, [p_1\, f(\tilde{y},\ x) + \sum_{2=1}^{m} p_j\, y_j - \sum_{i=1}^{n} w_i\, x_i] \tag{112a}$$

$$= \max_{x\ \tilde{y}}\, [p_1\, f(\tilde{y},\ x) + \tilde{p}\, \tilde{y} - w\, x] \text{ where}$$

$$\tilde{p} = (p_2,\, p_3,\, \ldots,\, p_m), \quad \tilde{y} = (y_2,\, y_3,\, \ldots,\, y_m),$$

$$w = (w_1,\, w_2,\, \ldots,\, w_n) \quad x = (x_1,\, x_2,\, \ldots,\, p_n)$$

If f($\tilde{y}$,x) is differentiable, then the first order conditions for maximizing profit are as follows.

$$\begin{aligned}\frac{\partial \pi}{\partial x_i} &= p_1 \frac{\partial f(\tilde{y},\, x)}{\partial x_i} - w_i = 0, \quad i = 1, 2, \ldots, n \\ \frac{\partial \pi}{\partial y_j} &= p_1 \frac{\partial f(\tilde{y},\, x)}{\partial y_j} + p_j = 0, \quad j = 2, 3, \ldots, m\end{aligned} \tag{113}$$

We can then obtain marginal products by rearranging the first order conditions. We can find the impact on y_1 of an increase in any of the inputs as

$$\frac{\partial f(\tilde{y},\, x)}{\partial x_i} = \frac{w_i}{p_1}, \quad i = 1, 2, \ldots, n \tag{114}$$

We can find the impact on y_1 of an increase in any of the other outputs as

$$\frac{\partial f(\tilde{y},\, x)}{\partial y_j} = -\frac{p_j}{p_1}, \quad j = 2, 3, \ldots, m \tag{115}$$

We can find other marginal products by solving the profit maximization problem with a different normalization than the one on y_1.

In the case of a single output we obtain

$$\frac{\partial f(x)}{\partial x_i} = \frac{w_i}{p}, \quad i = 1, 2, \ldots, n \tag{116}$$

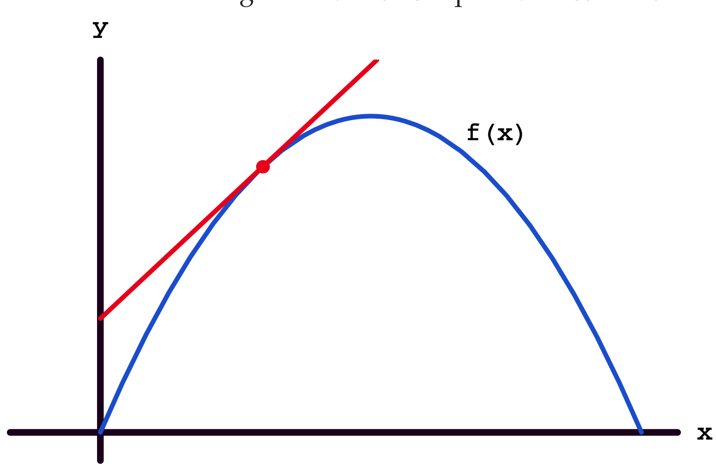
This is clear from figure 2. The isoprofit line is given by

$$\bar{\pi} = py - wx$$

Solving for y we obtain

$$y = \frac{\bar{\pi}}{p} + \frac{w}{p}x$$

FIGURE 2. Marginal Product is Equal to Price Ratio



10.2. **Output elasticity.** We normally think of output elasticity in terms of one output and many inputs so define it as follows.

$$\epsilon_i = \frac{\partial f(x)}{\partial x_i} \frac{x_i}{y} \tag{117}$$

If we substitute for $\frac{\partial f(x)}{\partial x_i}$ from equation 116 we obtain

$$\epsilon_i = \frac{w_i}{p} \frac{x_i}{y} \tag{118}$$

10.3. **Elasticity of scale.** Elasticity of scale is given by the sum of the output elasticities,i.e.,

$$\begin{aligned}\epsilon &= \sum_{i=1}^{n} \frac{\partial f(p,w)}{\partial x_i} \frac{x_i}{y} \\ &= \sum_{i=1}^{n} \frac{w_i}{p} \frac{x_i}{y} \\ &= \frac{\sum_{i=1}^{n} w_i \, x_i}{p \, y} \\ &= \frac{Cost}{Revenue}\end{aligned} \tag{119}$$

A profit maximizing firm will never produce in a region of the production function with increasing returns to scale assuming the technology eventually becomes locally concave and remains concave. As the result for a profit maximizing firm, returns to scale is just the ratio of cost to revenue.

11. Duality between the Profit Function and the Technology

We can show that if we have an arbitrarily specified profit function that satisfies the conditions in section 3, we can construct from it a reasonable technology that satisfies the conditions we specified in section 1.2.4 and that would generate the specified profit function. A general discussion of this duality is contained in Diewert [2]

11.1. **Constructing a technology from the profit function.** Let T* be defined as

$$T^* \equiv \{(x, y) : py - wx \leqq \pi(p, w), \forall (p, w) \geqq (0, 0)\} \tag{120}$$

To see the intuition of why we can construct T this way consider the case of one input and one output as in figure 3

If we pick a particular set of prices then T^* consists of all points below the line that is tangent to T(x,y) at the optimal input output combination. The equation $\{(x, y) : py - wx \leqq \pi(p, w)$ for a particular set of prices defines a line in R^2 or a hyperplane in R^{n+m}. Points below the line are considered to be in T^*. Now pick a different set of prices with w lower and construct a different line as in figure 4. The number of points in T^* is now less than before. If we add a third set of prices with w higher than in the original case as in figure 5, we can reduce the size of T^* even more. Continuing in this fashion we can recover T(x,y). This is an application of what is often called Minkowski's theorem.

To show that T is in fact equal to T^*, we need to show that $T^* \subseteq T$. We do this by assuming that some particular input output combination $(x^0, y^0) \in T^*$ but $(x^0, y^0) \notin T$. If $(x^0, y^0) \notin T$, then $(x^0,$

FIGURE 3. Half-Spaces and Technology Set

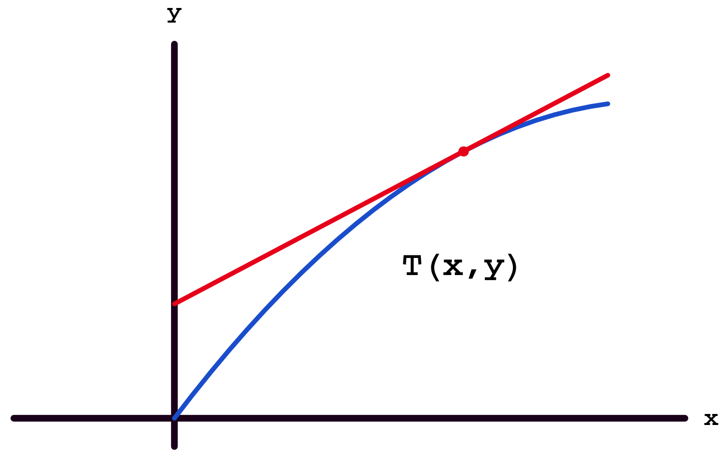
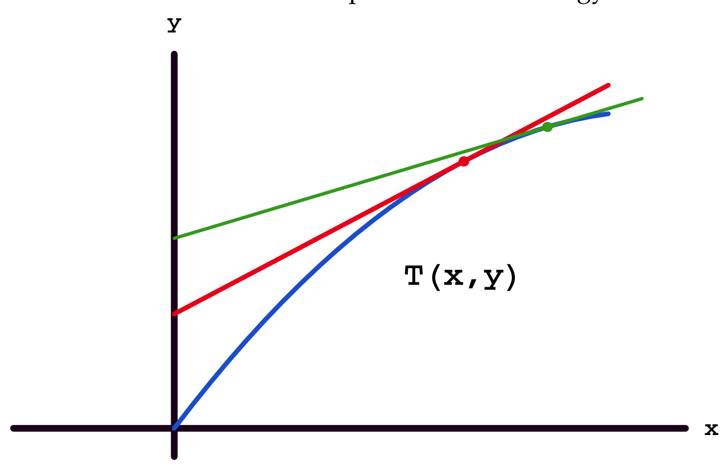


FIGURE 4. Two Half-Spaces and Technology Set



y^0) can be separated from T by a hyperplane (py^0 - wx^0). But T^* consists of points lying below the hyperplane defined by the maximal profits with prices p and w. This means that the point (x^0, y^0) cannot be in T^*. We can show this in figure 6. Consider point a in figure 6 which is not in T(x,y). It is in T^* if we only consider the half spaces defined by the hyperplanes intersecting T(x,y) at points c and d. But at the prices defined by a hyperplane with "slope" as the one passing through b, point a is not in T^*. If we consider any other point not in T, there exists a hyperplane that also excludes it from T^*. This implies than that $T^* \subseteq T$ and so $T^* \equiv T$. For more on this topic consult McFadden [12] or Fare and Primont [6].

11.2. **Using the properties of the profit function to recover properties of the technology.** We repeat here for convenience the properties of T(x,y) and π(p,w).

11.2.1. *Properties of the technology.*

FIGURE 5. Three Half-Spaces and Technology Set

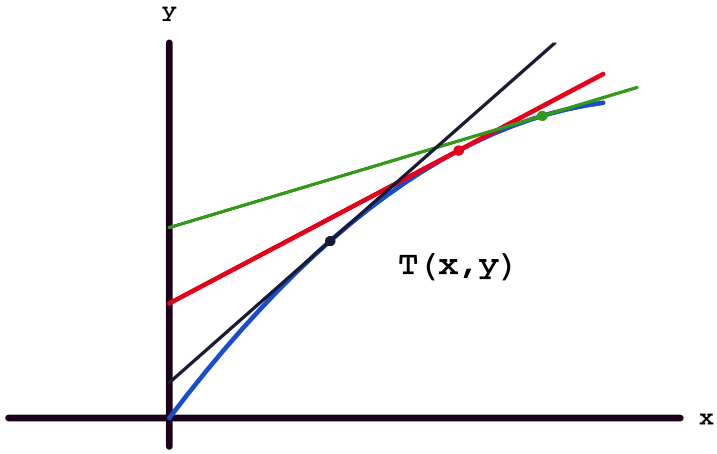
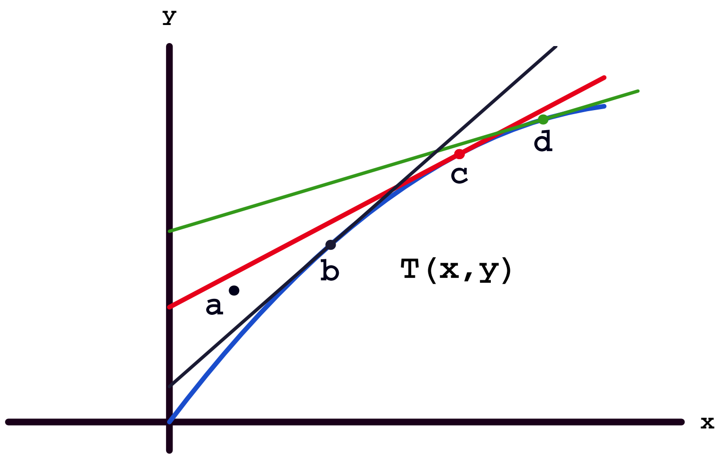


FIGURE 6. Separating Hyperplanes and the Technology Set



T.1a: Inaction and No Free Lunch. (0,y) ϵ T $\forall$ x ϵ R^n_+ and y ϵ R^m_+. This implies that T(x,y) is a non-empty subset of R^{m+n}_+

T.1b: (0,y) $\notin$ T, y $\geqq$ 0, y $\neq$ 0.

T.2: Input Disposability. If (x,y) ϵ T and $\theta \geq 1$ then (θx, y) ϵ T.

T.2.S: Strong Input Disposability. If (x,y) ϵ T and x′ $\geqq$ x, then (x′, y) ϵ T.

T.3: Output Disposability. $\forall$ (x,y) ϵ R^{n+m}_+ ,if (x, y) ϵ T and $0 < \lambda \leqq 1$ then (x, λy) ϵ T.

T.3.S: Strong Output Disposability. If (x,y) ϵ T and y′ $\leqq$ y, then (x, y′) ϵ T.

T.4: Boundedness. For every finite input vector x $\geqq$ 0, the set y ϵ P(x) is bounded from above. This implies that only finite amounts of output can be produced from finite amounts of inputs.

T.5: T(x) is a closed set. The assumption that P(x) and V(y) are closed does not imply that T is a closed set, so it is assumed. Specifically, if [$x^\ell \rightarrow x^0$, $y^\ell \rightarrow y^0$) and (x^ℓ, y^ℓ) ϵ T, $\forall$ ℓ] then (x^0 y^0) ϵ T.

T.9: T is a convex set. This is not implied by the convexity of P(x) and V(y). Specifically, a technology could exhibit a diminishing rate of technical substitution, a diminishing rate of product transformation and still exhibit increasing returns to scale.

11.2.2. *Properties of the profit function.*

$\pi.1$**:** π(p, w) is an extended real valued function (it can take on the value of $+\infty$ for finite prices) defined for all (p, w) $\geqq$ (0_m, 0_n)) and π(p, w) $\geq$ pa - wb for a fixed vector (a, b) $\geq$ (0_m, 0_n). This implies that π(p, w) $\geqq$ 0 if (0_m, 0_n) $\in$ T(x,y),which we normally assume.

$\pi.2$**:** π is nonincreasing in w

$\pi.3$**:** π is nondecreasing in p

$\pi.4$**:** π is a convex function

$\pi.5$**:** π is homogeneous of degree 1 in p and w.

We will consider only a few of the properties of T(x,y)

11.2.3. *T(x,y) is non-empty.* By π.1, π(p, w) $\geq$ pa - wb for a fixed vector (a, b) $\geq$ (0_m, 0_n) or π(p, w) $\geq$ 0 if (0_m, 0_n) $\in$ T(x,y). Given that π(p, w) $\geq$ 0, with x zero, there are obviously values of y (for example 0) which make (py - wx) less than π(p, w). So T^*(x,y) is not empty and (0,y) $\in$ T^*(x,y).

11.2.4. *Strong Input Disposability.* If (x,y) ϵ T and x′ $\geqq$ x, then (x′, y) ϵ T. .

Suppose that (x, y) ϵ T^*, 0 $\leq$ x $\leq$ x′ and $\forall$ (p, w) $\geqq$ 0. Then py - wx′ $\leq$ py - wx $\leq$ π(p, w) $\forall$ (p, w) $\geqq$ 0. Thus (y, x′) ϵ T^*.

11.2.5. *Strong Output Disposability.* If (x,y) ϵ T and y′ $\leqq$ y, then (x, y′) ϵ T. .

Suppose (y, x) ϵ T^* and 0 $\leq$ y′ $\leq$ y then py′ - wx $\leq$ py - wx $\leq$ π(p, w) $\forall$ (p, w) $\geqq$ 0. Thus (y′, x) ϵ T^*.

11.2.6. *T(x) is a closed and convex set.* From the definition, T* is the intersection of a family of closed half-spaces because we have a hyperplane in n+m space dividing the space. Thus T is a closed convex set (Rockafellar [13, p. 10].

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