# PROFIT MAXIMIZATION

## 1. DEFINITION OF A NEOCLASSICAL FIRM

A neoclassical firm is an organization that **controls** the transformation of **inputs** (resources it owns or purchases) into **outputs or products** (valued products that it sells) and **earns** the difference between what it receives in revenue and what it spends on inputs.

A technology is a description of process by which inputs are converted in outputs. There are a myriad of ways to describe a technology, but all of them in one way or another specify the outputs that are feasible with a given choice of inputs. Specifically, a production technology is a description of the set of outputs that can be produced by a given set of factors of production or inputs using a given method of production or production process.

We assume that neoclassical firms exist to make money. Such firms are called *for-profit* firms. We then set up the firm level decision problem as maximizing the net returns from the technologies controlled by the firm taking into account the demand for final consumption products, opportunities for buying and selling products from other firms, and the actions of other firms in the markets in which the firm participates. In perfectly competitive markets this means the firm will take prices as given and choose the levels of inputs and outputs that maximize profits. If the firm controls more than one production technology it takes into account the interactions between the technologies and the overall profits from the group of technologies. The profits (or net returns) to a particular production plan are given by the revenue obtained from the plan minus the costs of the inputs or

$$\pi = \Sigma_{j=1}^{m} p_j y_j - \Sigma_{i=1}^{n} w_i x_i \tag{1}$$

where $p_j$ is the price of the jth output and $w_i$ is the price of the ith input. In the case of a single output this can be written

$$\pi = p y - \Sigma_{i=1}^{n} w_i x_i \tag{2}$$

where p is the price of the single output y.

## 2. DESCRIPTIONS OF TECHNOLOGY

2.1. **Technology Sets.** A common way to describe a production technology is with a production set. The technology set for a given production process is defined as

$$T = \{(x, y) : x \, \epsilon \, R_+^n, \, y \, \epsilon \, R_+^m : x \text{ can produce y } \} \tag{3}$$

where x is a vector of inputs and y is a vector of outputs. The set consists of those combinations of x and y such that y can be produced from the given x.

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2.2. **Production Correspondence.** The output correspondence P, maps inputs $x \, \epsilon \, R_+^n$ into subsets of outputs, i.e., P: $R_+^n \to 2^{R_+^m}$, or $P(x) \subseteq R_+^m$. The set $P(x)$ is the set of all output vectors $y \, \epsilon \, R_+^m$ that are obtainable from the input vector $x \, \epsilon \, R_+^n$. We represent P in terms of the technology set as

$$P(x) \; = \; \{y : \; (x, \, y) \, \epsilon \, T\} \tag{4}$$

2.3. **Input Correspondence.** The input correspondence V, maps outputs $y \, \epsilon \, R_+^m$ into subsets of inputs, i.e., V: $R_+^m \to 2^{R_+^n}$, or $V(y) \subseteq R_+^n$. The set $V(y)$ is the set of all input vectors $x \, \epsilon \, R_+^n$ that are able to yield the output vector $y \, \epsilon \, R_+^m$. We represent V in terms of the technology set as

$$V(y) \; = \; \{x : \; (x, \, y) \, \epsilon \, T\} \tag{5}$$

2.4. **Relationships between representations: V(y), P(x) and T(x,y).** The technology set can be written in terms of either the input or output correspondence.

$$T \; = \; \{(x, \, y) : x \, \epsilon \, R_+^n, \; y \, \epsilon \, R_+^m, \text{such that x will produce y}\} \tag{6a}$$

$$T \; = \; \{(x, \, y) \, \epsilon \, R_+^{n+m} \; : y \, \epsilon P(x), \; x \epsilon \, R_+^n\} \tag{6b}$$

$$T \; = \; \{(x, \, y) \, \epsilon \, R_+^{n+m} \; : x \, \epsilon V(y), \; y \epsilon \, R_+^m\} \tag{6c}$$

We can summarize the relationships between the input correspondence, the output correspondence, and the production possibilities set in the following proposition.

**Proposition 1.** $y \, \epsilon \, P(x) \Leftrightarrow x \, \epsilon \, V(y) \Leftrightarrow (x,y) \, \epsilon \, T$

2.5. **Production Function.** In the case where there is a single output it is sometimes useful to represent the technology of the firm with a mathematical function that gives the maximum output attainable from a given vector of inputs. This function is called a production function and is defined as

$$\begin{aligned} f\left(x\right) \; &= \; \max_{y} \, \left[y : \; (x, \, y) \, \epsilon \, T\right] \\ &= \; \max_{y} \, \left[y : \; x \, \epsilon \, V(y)\right] \\ &= \; \max_{y \, \epsilon \, P(x)} \, \left[y\right] \end{aligned} \tag{7}$$

2.6. **Asymmetric Transformation Function.** In cases where there are multiple outputs one way to represent the technology of the firm is with an asymmetric transformation function. The function is asymmetric in the sense that it normalizes on one of the outputs, treating it asymmetrically with the other outputs. We usually normalize on the first output in the output vector, but this is not necessary. If $y = (y_1, y_2, \ldots, y_m)$ we can write it in the following asymmetric fashion $y = (y_1, \tilde{y})$ where $\tilde{y} = (y_2, y_3, \ldots, y_m)$. The transformation function is then defined as

$$\begin{aligned} f(\tilde{y}, \, x) \; &= \; \max_{y_1} \, \{y_1 : \; (y_1, \, \tilde{y}, \, x) \, \in \, T\}, \quad \text{if it exists} \\ &= \; -\infty \text{ otherwise}, \quad \tilde{y} \geq 0_{m-1}, \; x \geq 0_n \end{aligned} \tag{8}$$

This gives the maximum obtainable level of $y_1$, given levels of the other outputs and the input vector x. We could also define the asymmetric transformation function based on $y_i$. In this case it would give the maximum obtainable level of $y_i$, given levels of the other outputs (including $y_1$) and the input vector x. There are additional ways to describe multiproduct technologies using functions which will be discussed in a later section.

## 3. THE GENERAL PROFIT MAXIMIZATION PROBLEM

The general firm-level maximization problem can be written in a number of alternative ways.

$$\pi = \max_{x\,y} [\Sigma_{j=1}^{m}\, p_j y_j - \Sigma_{i=1}^{n}\, w_i\, x_i], \quad \text{such that } (x, y) \in T. \tag{9}$$

where T is represents the graph of the technology or the technology set. The problem can also be written as

$$\pi = \max_{x\,y} [\Sigma_{j=1}^{m}\, p_j y_j - \Sigma_{i=1}^{n}\, w_i\, x_i] \text{ such that } x \in V(y) \,] \tag{10a}$$

$$\pi = \max_{x\,y} [\Sigma_{j=1}^{m}\, p_j y_j - \Sigma_{i=1}^{n}\, w_i\, x_i] \text{ such that } y \in P(x) \,] \tag{10b}$$

where the technology is represented by V(y), the input requirement set, or P(x), the output set. We can write it in terms of functions in the following two ways

$$\pi = \max_{x} [p\, f(x_1, x_2, \ldots, x_n) - \Sigma_{i=1}^{n}\, w_i\, x_i] \tag{11a}$$

$$\pi = \max_{x\,\tilde{y}} [p_1\, f(\tilde{y}, x) + \Sigma_{2=1}^{m}\, p_j\, y_j - \Sigma_{i=1}^{n}\, w_i\, x_i] \tag{11b}$$

$$= \max_{x\,\tilde{y}} [p_1\, f(\tilde{y}, x) + \tilde{p}\, \tilde{y} - w\, x] \text{ where}$$

$$\tilde{p} = (p_2, p_3, \ldots, p_m), \quad \tilde{y} = (y_2, y_3, \ldots, y_m),$$
$$w = (w_1, w_2, \ldots, w_n) \quad x = (x_1, x_2, \ldots, p_n)$$

## 4. PROFIT MAXIMIZATION WITH A SINGLE OUTPUT AND A SINGLE INPUT

4.1. **Formulation of Problem.** The production function is given by

$$y = f(x) \tag{12}$$

If the production function is continuous and differentiable we can use calculus to obtain a set of conditions describing optimal input choice. If we let $\pi$ represent profit then we have

$$\pi = p\, f(x) - w\, x \tag{13}$$

If we differentiate the expression in equation 13 with respect to the input x obtain

$$\frac{\partial\, \pi}{\partial\, x} = p\, \frac{\partial f(x)}{\partial x} - w = 0 \tag{14}$$

Since the partial derivative of f with respect to x is the marginal product of x this can be interpreted as

$$p\, MP_x = w$$
$$\Rightarrow MVP_x = MFC_x \tag{15}$$

where $MVP_x$ is the marginal value product of x and $MFC_x$ (marginal factor cost) is its factor price. Thus the firm will continue using the input x until its marginal contribution to revenues just covers its costs. We can write equation 14 in an alternative useful way as follows

$$p \frac{\partial f(x)}{\partial x} - w = 0$$

$$\Rightarrow \frac{\partial f(x)}{\partial x} = \frac{w}{p}$$

(16)

This says that the slope of the production function is equal to the ratio of input price to output price. We can also view this as the slope of the isoprofit line. Remember that profit is given by

$$\pi = p\,y - w\,x$$

$$\Rightarrow y = \frac{\pi}{p} + \frac{w\,x}{p}$$

(17)

The slope of the line is then $\frac{w}{p}$. This relationship is demonstrated in figure 1.

FIGURE 1. Profit Maximization Point
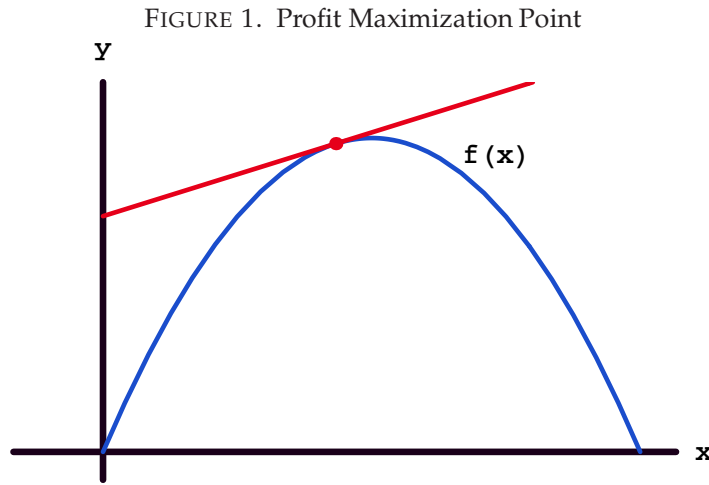


4.2. **Input demands.** If we solve equation 14 or equation 16 for x, we obtain the optimal value of x for a given p and w. As a function of w for a fixed p, this is the factor demand for x.

$$x^* = x(p,\,w)$$

(18)

4.3. **Sensitivity analysis.** We can investigate the properties of x(p,w) by substituting x(p,w) for x in equation 14 and then treating it as an identity.

$$p \frac{\partial f(x(p,w))}{\partial x} - w \equiv 0$$

(19)

If we differentiate equation 19 with respect to w we obtain

$$p \frac{\partial^2 f(x(p,w))}{\partial x^2} \frac{\partial x(p,w)}{\partial w} - 1 \equiv 0$$

(20)

As long as $\frac{\partial^2 f(x(p,w))}{\partial x^2} \neq 0$, we can write

$$\frac{\partial x(p,w)}{\partial w} \equiv \frac{1}{p \frac{\partial^2 f(x(p,w))}{\partial x^2}}$$

(21)

If the production function is concave then $\frac{\partial^2 f\,(\,x(p,w)\,)}{\partial x^2} \leq 0$. This then implies that $\frac{\partial\,x(p,w)}{\partial w} \leq 0$. Factor demand curves slope down.

If we differentiate equation 19 with respect to p we obtain

$$p\,\frac{\partial^2 f\,(\,x(p,w)\,)}{\partial x^2}\,\frac{\partial\,x(p,w)}{\partial p}\; +\; \frac{\partial f\,(\,x(p,w)\,)}{\partial x}\;\equiv\;0 \tag{22}$$

As long as $\frac{\partial^2 f\,(\,x(p,w)\,)}{\partial x^2}\;\neq\;0$, we can write

$$\frac{\partial\,x(p,w)}{\partial p}\;\equiv\;\frac{-\,\frac{\partial f\,(\,x(p,w)\,)}{\partial x}}{p\,\frac{\partial^2 f\,(\,x(p,w)\,)}{\partial x^2}} \tag{23}$$

If the production function is concave with a positive marginal product then $\frac{\partial\,x(p,w)}{\partial p}\;\geq\;0$. Factor demand rises with an increase in output price.

4.4. **Example.** Consider the production function given by

$$y\;=\;15\,x\;-\;.5\,x^2 \tag{24}$$

Now let the price of output be given by p = 5 and the price of the input be given by w = 10. The profit maximization problem can be written

$$\begin{aligned}
\pi\;&=\;\max_x\,[5\;f\,(x)\;-\;10\,x]\\
&=\;\max_x\,[5\,(15x\;-\;0.5x^2)\;-\;10\,x]\\
&=\;\max_x\,[65x\;-\;2.5x^2]
\end{aligned} \tag{25}$$

If we differentiate $\pi$ with respect to x we obtain

$$\begin{aligned}
65\;-\;5\,x\;&=0\\
\Rightarrow\;5\,x\;&=65\\
\Rightarrow\;x\;&=13
\end{aligned} \tag{26}$$

If we write this in terms of marginal value product and marginal factor cost we obtain

$$\begin{aligned}
5\,(15\;-\;x)\;-\;10\;&=0\\
\Rightarrow\;p\,MP_x\;-\;MFC_x\;&=0\\
\Rightarrow\;p\,MP_x\;&=MFC_x\\
\Rightarrow\;x\;&=13
\end{aligned} \tag{27}$$

## 5. PROFIT MAXIMIZATION WITH A SINGLE OUTPUT AND TWO INPUTS

5.1. **Formulation of Problem.** The production function is given by

$$y\;=\;f\,(x_1,\,x_2) \tag{28}$$

If the production function is continuous and differentiable we can use calculus to obtain a set of conditions describing optimal input choice. If we let $\pi$ represent profit then we have

$$\pi\;=\;p\,f\,(x_1,\,x_2)\;-\;w_1\,x_1\;-\;w_2\,x_2 \tag{29}$$

If we differentiate the expression in equation 29 with respect to each input we obtain

$$\frac{\partial \pi}{\partial x_1} = p \frac{\partial f(x_1, x_2)}{\partial x_1} - w_1 = 0$$

$$\frac{\partial \pi}{\partial x_2} = p \frac{\partial f(x_1, x_2)}{\partial x_2} - w_2 = 0$$

(30)

Since the partial derivative of f with respect to $x_j$ is the marginal product of $x_j$ this can be interpreted as

$$\pi = p f(x_1, x_2) - w_1 x_1 - w_2 x_2$$

(31)

$$p MP_1 = w_1$$

$$p MP_2 = w_2$$

$$\Rightarrow MVP_1 = MFC_1$$

$$\Rightarrow MVP_2 = MFC_2$$

(32)

where $MVP_j$ is the marginal value product of the jth input and $MFC_j$ (marginal factor cost) is its factor price.

5.2. **Second order conditions.** The second order conditions for a maxmimum in equation 29 are given by examining the Hessian of the objective function.

$$\nabla^2 \pi(x) = H_\pi = \begin{bmatrix} \frac{\partial^2 \pi(x^*)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_2 \partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \pi(x^*)}{\partial x_1^2} & \frac{\partial^2 \pi(x^*)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

(33)

Equation 29 has a **local maximum** at the point $x^*$ if $\frac{\partial^2 \pi}{\partial x_1^2}(x^*) < 0$ *and* $\frac{\partial^2 \pi}{\partial x_1^2} \frac{\partial^2 \pi}{\partial x_2^2} - \left[ \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \right]^2 > 0$ *at* $x^*$. We can say this in another way as follows. The leading principal minors of $\nabla^2 \pi(x)$ alternate in sign with the first leading principal minor being negative. Specifically,

$$\frac{\partial^2 \pi(x^*)}{\partial x_1^2} = p \frac{\partial^2 f(x^*)}{\partial x_1^2} < 0$$

$$\rightarrow \frac{\partial^2 f(x^*)}{\partial x_1^2} < 0, \text{ given that } p > 0$$

(34)

$$p \begin{vmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x^*)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x^*)}{\partial x_2^2} \end{vmatrix} > 0$$

5.3. **Input demands.** If we solve the equations in 30 for $x_1$ and $x_2$, we obtain the optimal values of x for a given p and w. As a function of w for a fixed p, this gives the vector of factor demands for x.

$$x^* = x(p, w_1, w_2) = (x_1(p, w_1, w_2), x_2(p, w_1, w_2))$$

(35)

5.4. **Homogeneity of degree zero of input demands.** Consider the profit maximization problem if we multiply all prices (p $w_1$, $w_2$) by a constant $\lambda$ as follows

$$\pi = p f(x_1, x_2) - w_1 x_1 - w_2 x_2 \tag{36a}$$

$$\Rightarrow \pi(\lambda) = \lambda p f(x_1, x_2) - \lambda w_1 x_1 - \lambda w_2 x_2 \tag{36b}$$

$$= \lambda [p f(x_1, x_2) - w_1 x_1 - w_2 x_2] \tag{36c}$$

Maximizing 36a or 36c will give the same results for $x(p, w_1, w_2)$ because $\lambda$ is just a constant that will not affect the optimal choice.

5.5. **Sensitivity analysis.**

5.5.1. *Response of factor demand to input prices.* We can investigate the properties of $x(p, w_1, w_2)$ by substituting $x(p, w_1, w_2)$ for x in equation 30 and then treating it as an identity.

$$p \frac{\partial f(x(p, w_1, w_2))}{\partial x_1} \equiv w_1$$
$$p \frac{\partial f(x(p, w_1, w_2))}{\partial x_2} \equiv w_2 \tag{37}$$

If we differentiate equation 37 with respect to $w_1$ we obtain

$$p \frac{\partial^2 f(x(p, w_1, w_2))}{\partial x_1^2} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} + p \frac{\partial^2 f(x(p, w_1, w_2))}{\partial x_2 \partial x_1} \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} \equiv 1$$
$$p \frac{\partial^2 f(x(p, w_1, w_2))}{\partial x_1 \partial x_2} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} + p \frac{\partial^2 f(x(p, w_1, w_2))}{\partial x_2^2} \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} \equiv 0 \tag{38}$$

We can write this in more abbreviated notation as

$$p f_{11} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} + p f_{12} \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} \equiv 1$$
$$p f_{21} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} + p f_{22} \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} \equiv 0 \tag{39}$$

If we differentiate equation 37 with respect to $w_2$ we obtain

$$p f_{11} \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} + p f_{12} \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \equiv 0$$
$$p f_{21} \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} + p f_{22} \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \equiv 1 \tag{40}$$

Now write equations 39 and 40 in matrix form.

$$p \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{41}$$

If the Hessian matrix is non-singular, we can solve this equation for the matrix of first derivatives,

$$\begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}^{-1} \tag{42}$$

We can compute the various partial derivatives on the left hand side of equation 42 by inverting the Hessian or using Cramer's rule in connection with equation 41.

The inverse of the Hessian of a two variable production function gan be computed by using the adjoint. The adjoint is the transpose of the cofactor matrix of the Hessian. For a square nonsingular matrix A, its inverse is given by

$$A^{-1} = \frac{1}{|A|} A^{+} \tag{43}$$

We compute the inverse by first computing the cofactor matrix.

$$\nabla^2 f(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

$$cofactor[f_{11}] = (-1)^2 f_{22}$$

$$cofactor[f_{12}] = (-1)^3 f_{21}$$

$$cofactor[f_{21}] = (-1)^3 f_{12} \tag{44}$$

$$cofactor[f_{22}] = (-1)^4 f_{11}$$

$$\Rightarrow cofactor\left[\nabla^2 f(x_1, x_2, \ldots, x_n)\right] = \begin{pmatrix} f_{22} & -f_{21} \\ -f_{12} & f_{11} \end{pmatrix}$$

We then find the adjoint by taking the transpose of the cofactor matrix.

$$adjoint\left[\nabla^2 f(x_1, x_2, \ldots, x_n)\right] = \begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix} \tag{45}$$

We obtain the inverse by dividing the adjoint by the determinant of $\left[\nabla^2 f(x_1, x_2, \ldots, x_n)\right]$.

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}}{f_{11} f_{22} - f_{12} f_{21}} \tag{46}$$

Referring back to equation 42, we can compute the various partial derivatives.

$$\begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \end{pmatrix} = \frac{\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}}{p(f_{11} f_{22} - f_{12} f_{21})} \tag{47}$$

We then obtain

$$\frac{\partial x_1(p, w_1, w_2)}{\partial w_1} = \frac{f_{22}}{p(f_{11} f_{22} - f_{12}^2)} \tag{48a}$$

$$\frac{\partial x_1(p, w_1, w_2)}{\partial w_2} = \frac{-f_{12}}{p(f_{11} f_{22} - f_{12}^2)} \tag{48b}$$

$$\frac{\partial x_2(p, w_1, w_2)}{\partial w_1} = \frac{-f_{21}}{p(f_{11} f_{22} - f_{12}^2)} \tag{48c}$$

$$\frac{\partial x_2(p, w_1, w_2)}{\partial w_2} = \frac{f_{11}}{p(f_{11} f_{22} - f_{12}^2)} \tag{48d}$$

The denominator is positive by second order conditions in eqaution 34 or the fact that f( ) is concave. Because we have a maximum, $f_{11}$ and $f_{22}$ are less than zero. Own price derivatives are negative and so factor demand curves slope downwards. The sign of the cross partial derivatives depends on the sign of $f_{12}$. If $x_1$ and $x_2$ are gross substitutes, then $f_{12}$ is negative and the second cross partials are positive. This means that the demand for $x_1$ goes up as the price of $x_2$ goes up.

5.5.2. *Finding factor demand response using Cramer's rule.* When we differentiate the first order conditions we obtain

$$p \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_1(p, w_1, w_2)}{\partial w_2} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} & \frac{\partial x_2(p, w_1, w_2)}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial w_1} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial w_1} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{p} \\ 0 \end{pmatrix} \tag{49}$$

To find $\frac{\partial x_1(p, w_1, w_2)}{\partial w_1}$ we replace the first column of the Hessian with the right hand side vector and then form the ratio of the determinant of this matrix to the determinant of the Hessian. First for the determinant of the matrix with the righthand side replacing the first column.

$$\begin{vmatrix} \frac{1}{p} & f_{12} \\ 0 & f_{22} \end{vmatrix} = \frac{1}{p} f_{22} \tag{50}$$

Then find the determinant of the Hessian

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21}0 & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{12} f_{21} \tag{51}$$

Forming the ratio we obtain

$$\frac{\partial x_1(p, w_1, w_2)}{\partial w_1} = \frac{f_{22}}{p(f_{11} f_{22} - f_{12} f_{21})} \tag{52}$$

which is the same as in equation 48.

5.5.3. *Response of factor demand to output price.* If we differentiate equation 37 with respect to p we obtain

$$p f_{11} \frac{\partial x_1(p, w_1, w_2)}{\partial p} + p f_{12} \frac{\partial x_2(p, w_1, w_2)}{\partial p} \equiv -f_1$$

$$p f_{21} \frac{\partial x_1(p, w_1, w_2)}{\partial p} + p f_{22} \frac{\partial x_2(p, w_1, w_2)}{\partial p} \equiv -f_2$$

(53)

Now write equation 53 in matrix form.

$$p \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial p} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial p} \end{pmatrix} = \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix}$$

(54)

Multiply both sides of equation 54 by $\frac{1}{p} \nabla^2 f(x_1, x_2, \ldots, x_n)^{-1}$ to obtain

$$\begin{pmatrix} \frac{\partial x_1(p, w_1, w_2)}{\partial p} \\ \frac{\partial x_2(p, w_1, w_2)}{\partial p} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}^{-1} \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}}{p (f_{11} f_{22} - f_{12} f_{21})} \begin{pmatrix} -f_1 \\ -f_2 \end{pmatrix}$$

(55)

$$= \frac{\begin{pmatrix} -f_1 f_{22} + f_2 f_{12} \\ f_1 f_{21} - f_2 f_{11} \end{pmatrix}}{p (f_{11} f_{22} - f_{12} f_{21})}$$

This implies that

$$\frac{\partial x_1(p, w_1, w_2)}{\partial p} = \frac{-f_1 f_{22} + f_2 f_{12}}{p (f_{11} f_{22} - f_{12}^2)}$$

$$\frac{\partial x_2(p, w_1, w_2)}{\partial p} = \frac{f_1 f_{21} - f_2 f_{11}}{p (f_{11} f_{22} - f_{12}^2)}$$

(56)

These derivatives can be of either sign. Inputs usually have a positive derivative with respect to output price.

5.6. **Example.**

5.6.1. *Production function.* Consider the following production function.

$$y = f(x_1, x_2)$$
$$= 30x_1 + 16x_2 - x_1^2 + x_1 x_2 - 2 x_2^2$$

(57)

The first and second partial derivatives are given by

$$\frac{\partial f(x_1,\, x_2)}{\partial x_1} = 30 \,-\, 2\,x_1 \,+\, x_2$$

$$\frac{\partial f(x_1,\, x_2)}{\partial x_2} = 16 \,+\, x_1 \,-\, 4\,x_2$$

$$\frac{\partial^2 f(x_1,\, x_2)}{\partial x_1^2} = -\,2 \tag{58}$$

$$\frac{\partial^2 f(x_1,\, x_2)}{\partial x_1 \partial x_2} = 1$$

$$\frac{\partial^2 f(x_1,\, x_2)}{\partial x_2^2} = -\,4$$

The Hessian is

$$\nabla^2 f(x_1,\, x_2) \;=\; \begin{pmatrix} \frac{\partial^2 f(x_1,\, x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,\, x_2)}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f(x_1,\, x_2)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x_1,\, x_2)}{\partial x_2^2} \end{pmatrix} \;=\; \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix} \tag{59}$$

The determinant of the Hessian is given by

$$\begin{vmatrix} -2 & 1 \\ 1 & -4 \end{vmatrix} \;=\; 8 \,-\, 1 \;=\; 7 \tag{60}$$

5.6.2. *Profit maximization.* Profit is given by

$$\begin{aligned} \pi &= p\,f(x_1,\, x_2) \,-\, w_1\,x_1 \,-\, w_2\,x_2 \\ &= p\left[\,30x_1 \,+\, 16x_2 \,-\, x_1^2 \,+\, x_1\,x_2 \,-\, 2\,x_2^2\,\right] \,-\, w_1\,x_1 \,-\, w_2\,x_2 \end{aligned} \tag{61}$$

We maximize profit by taking the derivatives of 61 setting them equal to zero and solving for $x_1$ and $x_2$.

$$\frac{\partial \pi}{\partial x_1} = p\,[\,30 \,-\, 2\,x_1 \,+\, x_2\,] \,-\, w_1 \;=\; 0$$

$$\frac{\partial \pi}{\partial x_2} = p\,[\,16 \,+\, x_1 \,-\, 4\,x_2\,] \,-\, w_2 \;=\; 0 \tag{62}$$

Rearranging 62 we obtain

$$30 \,-\, 2\,x_1 \,+\, x_2 \;=\; \frac{w_1}{p} \tag{63a}$$

$$16 \,+\, x_1 \,-\, 4\,x_2 \;=\; \frac{w_2}{p} \tag{63b}$$

Multiply equation 63b by 2 and add them together to obtain

$$62 - 7\,x_2 = \frac{w_1}{p} + 2\,\frac{w_2}{p}$$

$$\Rightarrow 7\,x_2 = 62 - \frac{w_1}{p} - 2\,\frac{w_2}{p} \tag{64}$$

$$\Rightarrow x_2 = \frac{62}{7} - \frac{w_1}{7p} - \frac{2w_2}{7p}$$

Now substitute $x_2$ in equation 63b and solve for $x_1$.

$$16 + x_1 - 4\left(\frac{62}{7} - \frac{w_1}{7p} - \frac{2w_2}{7p}\right) = \frac{w_2}{p}$$

$$\Rightarrow x_1 = \frac{w_2}{p} + 4\left(\frac{62}{7} - \frac{w_1}{7p} - \frac{2w_2}{7p}\right) - 16 \tag{65}$$

$$= \frac{w_2}{p} + \frac{136}{7} - \frac{4w_1}{7p} - \frac{8w_2}{7p}$$

$$= \frac{136}{7} - \frac{4w_1}{7p} - \frac{w_2}{7p}$$

5.6.3. *Necessary and sufficient conditions for a maximum.* Consider the Hessian of the profit equation.

$$\nabla^2\,\pi(x_1,\,x_2) = \begin{pmatrix} \frac{\partial^2\pi(x_1,\,x_2)}{\partial x_1^2} & \frac{\partial^2\pi(x_1,\,x_2)}{\partial x_2\partial x_1} \\ \frac{\partial^2\pi(x_1,\,x_2)}{\partial x_1\partial x_2} & \frac{\partial^2\pi(x_1,\,x_2)}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} -2p & p \\ p & -4p \end{pmatrix} \tag{66}$$

For a maximum we need the diagonal elements to be negative and the determinant to be positive. The diagonal elements are negative. The determinant of the Hessian is given by

$$\begin{vmatrix} -2p & p \\ p & -4p \end{vmatrix} = 8p^2 - p^2 = 7p^2 \geq 0 \tag{67}$$

5.6.4. *Input demand derivatives computed via first order conditions and formulas.* Consider the change in input demand with a change in input price. Remember the Hessian of the production function from equation 59 is given by

$$\nabla^2\,f(x_1,\,x_2) = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}$$

$$\begin{vmatrix} -2 & 1 \\ 1 & -4 \end{vmatrix} = 7$$

Now substitute in the formulas for input demand derivatives from equation 48

$$\frac{\partial x_1(p,\, w_1,\, w_2)}{\partial w_1} \;=\; \frac{f_{22}}{p\,(\,f_{11}\,f_{22}\,-\,f_{12}^2\,)} \;=\; \frac{-4}{7p} \tag{68a}$$

$$\frac{\partial x_1(p,\, w_1,\, w_2)}{\partial w_2} \;=\; \frac{-f_{12}}{p\,(\,f_{11}\,f_{22}\,-\,f_{12}^2\,)} \;=\; \frac{-1}{7p} \tag{68b}$$

$$\frac{\partial x_2(p,\, w_1,\, w_2)}{\partial w_1} \;=\; \frac{-f_{21}}{p\,(\,f_{11}\,f_{22}\,-\,f_{12}^2\,)} \;=\; \frac{-1}{7p} \tag{68c}$$

$$\frac{\partial x_2(p,\, w_1,\, w_2)}{\partial w_2} \;=\; \frac{f_{11}}{p\,(\,f_{11}\,f_{22}\,-\,f_{12}^2\,)} \;=\; \frac{-2}{7p} \tag{68d}$$

5.6.5. *Input demand derivatives computed from the optimal input demand equations.*

$$x_1 \;=\; \frac{136}{7} \;-\; \frac{4w_1}{7p} \;-\; \frac{w_2}{7p}$$

$$x_2 \;=\; \frac{62}{7} \;-\; \frac{w_1}{7p} \;-\; \frac{2w_2}{7p}$$

$$\frac{\partial x_1}{\partial w_1} \;=\; -\,\frac{4}{7p}$$

$$\frac{\partial x_1}{\partial w_2} \;=\; -\,\frac{1}{7p} \tag{69}$$

$$\frac{\partial x_2}{\partial w_1} \;=\; -\,\frac{1}{7p}$$

$$\frac{\partial x_2}{\partial w_2} \;=\; -\,\frac{2}{7p}$$

## 6. Profit Maximization with a Single Output and Multiple Inputs

### 6.1. **Formulation of Problem.** The production function is given by

$$y \;=\; f\,(x_1,\, x_2,\, \cdots,\, x_n) \tag{70}$$

If the production function is continuous and differentiable we can use calculus to obtain a set of conditions describing optimal input choice. If we let $\pi$ represent profit then we have

$$\pi \;=\; p\,f\,(x_1,\, x_2,\, \cdots,\, x_n) \;-\; \sum_{j=1}^{n} w_i\, x_i, \quad j \,=\, 1,\, 2,\, \cdots n \tag{71}$$

If we differentiate the expression in equation 71 with respect to each input we obtain

$$\frac{\partial \pi}{\partial x_j} \;=\; p\,\frac{\partial f\,(x)}{\partial x_j} \;-\; w_j \;=\; 0, \quad j \,=\, 1,\, 2,\, \cdots n \tag{72}$$

Since the partial derivative of f with respect to $x_j$ is the marginal product of $x_j$ this can be interpreted as

$$MP_j \;= w_j,\, j \,=\, 1,\, 2,\, \ldots\, n$$
$$\Rightarrow\; MVP_j \;= MFC_j,\, j \,=\, 1,\, 2,\, \ldots\, n \tag{73}$$

where $\text{MVP}_j$ is the marginal value product of the jth input and $\text{MFC}_j$ (marginal factor cost) is its factor price.

6.2. **Input demands.** If we solve the equations in 72 for $x_j$, j = 1, 2, ..., n, we obtain the optimal values of x for a given p and w. As a function of w for a fixed p, this gives the vector of factor demands for x.

$$x^* \; = \; x(p, \, w) \; = \; (x_1(p, w), x_2(p, w), \ldots, x_n(p, w)) \tag{74}$$

6.3. **Second order conditions.** The second order conditions for a maxmimum in equation 71 are given by examining the Hessian of the objective function.

$$\nabla^2 \pi(x) \; = \; H_\pi \; = \; \begin{bmatrix} \frac{\partial^2 \pi(x^*)}{\partial x_1 \, \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_1 \, \partial x_2} & \cdots & \frac{\partial^2 \pi(x^*)}{\partial x_1 \, \partial x_n} \\ \frac{\partial^2 \pi(x^*)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_2 \, \partial x_2} & \cdots & \frac{\partial^2 \pi(x^*)}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 \pi(x^*)}{\partial x_n \, \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_n \, \partial x_2} & \cdots & \frac{\partial^2 \pi(x^*)}{\partial x_n \, \partial x_n} \end{bmatrix} \; = \; \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{n1} & \pi_{n2} & \cdots & \pi_{nn} \end{bmatrix} \tag{75}$$

Equation 71 has a **local maximum** at the point $x^*$ if the leading principal minors of $\nabla^2 \pi(x)$ alternate in sign with the first leading principal minor being negative, the second positive and so forth. Thus $\pi_{11} \; < \; 0$ and $\pi_{11}\pi_{22} \; - \; \pi_{12}\pi_{21} \; > \; 0$ and so on. Specifically,

$$\frac{\partial^2 \pi(x^*)}{\partial x_1^2} \; = \; p \, \frac{\partial^2 f(x^*)}{\partial x_1^2} \; < 0$$

$$\rightarrow \; \frac{\partial^2 f(x^*)}{\partial x_1^2} \; < 0, \text{ given that } p \; > \; 0$$

$$p \begin{vmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2 f(x^*)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f(x^*)}{\partial x_2^2} \end{vmatrix} \; > 0 \tag{76}$$

$$p \begin{vmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_2} & \frac{\partial^2 f(x^*)}{\partial x_1 \, \partial x_3} \\ \frac{\partial^2 f(x^*)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f(x^*)}{\partial x_2^2} & \frac{\partial^2 f(x^*)}{\partial x_2 \, \partial x_3} \\ \frac{\partial^2 f(x^*)}{\partial x_3 \, \partial x_1} & \frac{\partial^2 f(x^*)}{\partial x_3 \partial x_2} & \frac{\partial^2 f(x^*)}{\partial x_3^2} \end{vmatrix} \; < 0$$

$$\vdots$$

6.4. **Sensitivity analysis.** We can investigate the properties of x(p,w) by substituting x(p,w) for x in equation 72 and then treating it as an identity.

$$p \, \frac{\partial f \left( \, x(p, w) \, \right)}{\partial x_j} \; - \; w_j \; \equiv \; 0, \quad j \; = \; 1, \, 2, \, \cdots n \tag{77}$$

If we differentiate the first equation in 77 with respect to $w_j$ we obtain

$$p\frac{\partial^2 f(x(p,\ w))}{\partial x_1^2}\frac{\partial x_1(p,\ w)}{\partial w_j} + p\frac{\partial^2 f(x(p,w)}{\partial x_2\,\partial x_1}\frac{\partial x_2(p,w)}{\partial w_j} + p\frac{\partial^2 f(x(p,w)}{\partial x_3\partial x_1}\frac{\partial\,x_3(p,\ w)}{\partial w_j} + \cdots \equiv 0$$

$$\Rightarrow \left( p\frac{\partial^2 f(x(p,w))}{\partial x_1^2} \quad p\frac{\partial^2 f(x(p,w)}{\partial x_2\,\partial x_1} \quad p\frac{\partial^2 f(x(p,w)}{\partial x_3\partial x_1} \cdots \frac{\partial^2 f(x(p,w)}{\partial x_n\partial x_1} \right) \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w_j} \\ \frac{\partial x_2(p,w)}{\partial w_j} \\ \frac{\partial x_3(p,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(p,w)}{\partial w_j} \end{pmatrix} \equiv 0 \tag{78}$$

If we differentiate the second equation in 77 with respect to $w_j$ we obtain

$$p\frac{\partial^2 f(x(p,\ w))}{\partial x_1\partial x_2}\frac{\partial x_1(p,\ w)}{\partial w_j} + p\frac{\partial^2 f(x(p,w)}{\partial x_2^2}\frac{\partial x_2(p,w)}{\partial w_j} + p\frac{\partial^2 f(x(p,w)}{\partial x_3\partial x_2}\frac{\partial\,x_3(p,\ w)}{\partial w_j} + \cdots \equiv 0$$

$$\Rightarrow \left( p\frac{\partial^2 f(x(p,w))}{\partial x_1\partial x_2} \quad p\frac{\partial^2 f(x(p,w)}{\partial x_2^2} \quad p\frac{\partial^2 f(x(p,w)}{\partial x_3\partial x_2} \cdots \frac{\partial^2 f(x(p,w)}{\partial x_n\partial x_2} \right) \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w_j} \\ \frac{\partial x_2(p,w)}{\partial w_j} \\ \frac{\partial x_3(p,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(p,w)}{\partial w_j} \end{pmatrix} \equiv 0 \tag{79}$$

If we differentiate the j$^{\text{th}}$ equation in 77 with respect to $w_j$ we obtain

$$p\frac{\partial^2 f(x(p,\ w))}{\partial x_1\partial x_j}\frac{\partial x_1(p,\ w)}{\partial w_j} + p\frac{\partial^2 f(x(p,w)}{\partial x_2\partial x_j}\frac{\partial x_2(p,w)}{\partial w_j} + \cdots + p\frac{\partial^2 f(x(p,w)}{\partial x_j^2}\frac{\partial\,x_j(p,\ w)}{\partial w_j} + \cdots \equiv 1$$

$$\Rightarrow \left( p\frac{\partial^2 f(x(p,w))}{\partial x_1\partial x_j} \quad p\frac{\partial^2 f(x(p,w)}{\partial x_2\partial x_j} \quad \cdots \quad p\frac{\partial^2 f(x(p,w)}{\partial x_j^2} \cdots \frac{\partial^2 f(x(p,w)}{\partial x_n\partial x_2} \right) \begin{pmatrix} \frac{\partial x_1(p,\ w)}{\partial w_j} \\ \frac{\partial x_2(p,\ w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_j(p,\ w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(p,\ w)}{\partial w_j} \end{pmatrix} \equiv 1$$

$$\tag{80}$$

Continuing in the same fashion we obtain

$$p \begin{pmatrix} \frac{\partial^2 f(x(p,w))}{\partial x_1^2} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \, \partial x_1} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_j \partial x_1} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_1} \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(p,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_j \partial x_2} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_j^2} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_j \partial x_n} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n^2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w_j} \\ \frac{\partial x_2(p,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_j(p,w)}{\partial w_j} \\ \vdots \\ \frac{\partial x_n(p,w)}{\partial w_j} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{81}$$

If we then consider derivatives with respect to each of the $w_j$ we obtain

$$p \begin{pmatrix} \frac{\partial^2 f(x(p,w))}{\partial x_1^2} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \, \partial x_1} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_1} \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(p,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_j} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n^2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1(p,\,w)}{\partial w_1} & \frac{\partial x_1(p,\,w)}{\partial w_2} & \cdots & \frac{\partial x_1(p,\,w)}{\partial w_n} \\ \frac{\partial x_2(p,\,w)}{\partial w_1} & \frac{\partial x_2(p,\,w)}{\partial w_2} & \cdots & \frac{\partial x_2(p,\,w)}{\partial w_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n(p,\,w)}{\partial w_1} & \frac{\partial x_n(p,\,w)}{\partial w_2} & \cdots & \frac{\partial x_n(p,\,w)}{\partial w_n} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\tag{82}$$

We can then write

$$\begin{pmatrix} \frac{\partial x_1(p,w)}{\partial w_1} & \frac{\partial x_1(p,w)}{\partial w_2} & \cdots & \frac{\partial x_1(p,w)}{\partial w_n} \\ \frac{\partial x_2(p,w)}{\partial w_1} & \frac{\partial x_2(p,w)}{\partial w_2} & \cdots & \frac{\partial x_2(p,w)}{\partial w_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n(p,w)}{\partial w_1} & \frac{\partial x_n(p,w)}{\partial w_2} & \cdots & \frac{\partial x_n(p,w)}{\partial w_n} \end{pmatrix} \equiv \frac{1}{p} \begin{pmatrix} \frac{\partial^2 f(x(p,w))}{\partial x_1^2} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \, \partial x_1} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_1} \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x(p,w)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_j} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_j} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n \partial x_j} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 f(x(p,w))}{\partial x_1 \partial x_n} & \frac{\partial^2 f(x(p,w)}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f(x(p,w)}{\partial x_n^2} \end{pmatrix}^{-1} \tag{83}$$

If the production function is concave, the Hessian will be at least negative semidefinite. This means that its characteristics roots are all negative or zero. If the Hessian is negative definite then its characteristics roots are all negative. A negative definite matrix is invertible. The inverse of an invertible matrix has characteristics roots which are reciprocals of the characteristic roots of the original matrix. So if the roots of the original matrix are all negative, the roots of the inverse will also be negative. If the characteristics roots of a matrix are all negative then the matrix is negative definite. And the diagonal elements of negative definite matrix are all negative. So own price derivatives are negative. The second order conditions for profit maximization also imply that the Hessian is negative definite as is clear from equation 76.