DECISION MAKING WITH UNCERTAINTY AND RISK AVERSION

1. INTRODUCTION

1.1. **The underlying idea of decision making under uncertainty.** We are interested in how a decision maker chooses among alternative courses of action when the consequences of each action are not know at the time the choice is made. Individuals may make different choices in a setting involving uncertainty than they will in one where outcomes are known. These differences are usually attributed to "risk preferences"

1.2. **Underlying framework for the problem.**
1. There are a number of outcomes for the decision problem. They are represented by a non-empty set of prizes or things that matter to the decision maker which is denoted X.
2. There are consequences which are represented by a non-empty set C. Consequences can be anything that has to do with the welfare of a decision maker. C can be a probability space over a set of outcomes or an outcome. One outcome might be you get a box with 3 oranges and 2 apples inside. Another might be you get two Powerball tickets purchased 3 December.
3. Feasible acts are a non-empty set denoted by A_0
4. The set of *conceivable acts* denoted by A contains the set of feasible acts. For example there may be no available action that leads to wining the lottery with certainty. This action is conceivable but not feasible.
5. A mapping from the elements of A_0 to subsets of C. For example, choosing to take curtain number 1 on "Let's Make a Deal" gives you some of the prizes that are available that day. Ultimately each act will result in a unique element of C, but which element occurs in not known *a priori*.
6. A state of nature is a function that assigns to every feasible act a consequence from the set of consequences corresponding to this act. For example, the consequences from raising the price of a product you sell might be that profits increase, profits decrease, or profits remain the same. State of nature "one" might be that profits decrease. The set of all states of nature is denoted by S.
7. Actions can be considered to be a mapping from the set of states to the set of consequences.
8. Constant acts are those which give the same consequence in all states of the world.
9. Risk is a situation in which the set of states is a singleton or all acts are constants. Consequences in this framework consist of probability measures or lotteries on a set of outcomes. For example, if the set of states is a singleton, an act represents the choosing a particular lottery or probability measure on a set of outcomes. Consider a gambler who is faced with two

possible slot machines to play. The first machine gives a payoff of $-1.00 with probability .9, a payoff of $4.00 with probability .05 and a payoff of $100.00 with probability .05. The second machine gives a payoff of $-1.00 with probability .8, a payoff of $4.00 with probability .16, and $100.00 with a probability of .04. The outcomes there are (-1.00, 4.00, 100.00). Each act induces a different lottery on the outcomes. The state of the world, the existence of the slot machines, and the associated lotteries, is a constant.

10. Uncertainty is a situation in which the set of consequences, C, coincides with the set of outcomes, X. The set of acts, A, consists of all functions from the set of states, S, to X. A preference relation on A is a primitive of the model. In this set-up there are no objective probabilities (probability model), but subjective probabilities are developed as part of the decision problem.

1.3. **Preference relations.** A preference relation is a binary relation, $\succeq$, on A that is

1. complete - for all a, b ϵ A either a $\succeq$ b or b $\succeq$ a
2. transitive - for all a,b,c ϵ A, a $\succeq$ b and b $\succeq$ c imply a $\succeq$

1.4. **Representing the preference relation.** A real valued function U on A represents $\succeq$ is for all a, b ϵA, a $\succeq$ b iff U(a) $\succeq$ U(b). The function U is called the utility function.

The most common way to represent preferences in such models is with a representation functional that is the sum of the products of utilities and probabilities of outcomes.

2. Expected utility theory (von Neumann Morgenstern)

For the analysis in this section, assume the the set of consequences C is finite.

2.1. **Lotteries.**

2.1.1. *Definition of a simple lottery.* A simple lottery L is a list L = $(p_1, p_2, \ldots, p_N)$ with $p_n \geq 0$ for all n and $\Sigma_{n=1}^{N} p_n = 1$ where p_n is interpreted as the probability of outcome n occurring. A simple lottery can be represented geometrically as a point in an N or (N-1) dimensional simplex, $\Delta = p \in R_+^N : p_1 + p_2 + ... + p_N = 1$. Consider the simple lottery represented in figure 1.

Each point in the simplex represents a particular lottery which yields consequence x_1 with probability p_1 etc. When N = 3 it is convenient to use a two dimensional diagram in the form of an equilateral triangle with altitude equal to one. This is convenient geometrically because the length of a side in this case is equal to $\frac{2}{\sqrt{3}}$ and the sum of the perpendiculars from any point to the three sides is equal to 1. For example at a vertex (probability mass equal to one at that point) the length to the opposite side is equal to the altitude of 1. Similarly a point at the center of the triangle has length of 1/3 to each side. Or a point midway between two endpoints along a side has length $\frac{1}{2}$ to the other two sides. The two dimensional representation of the lottery in figure 1 is contained in figure 2.

FIGURE 1. Simplex Representing a Simple Lottery
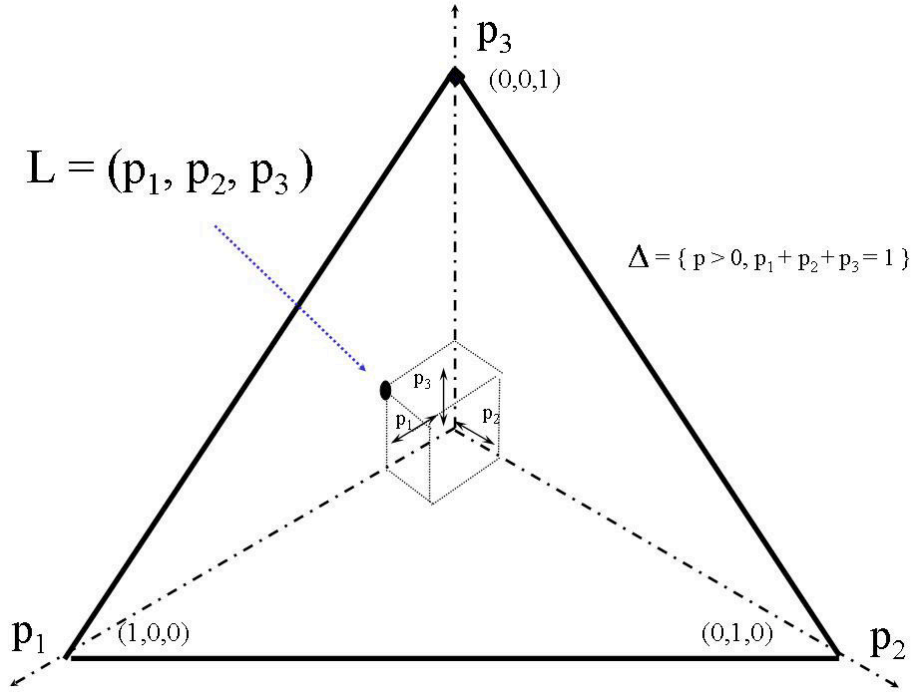
2.1.2. *Definition of a compound lottery.* Given K simple lotteries $L_k = (p_1^k, p_2^k, ..., p_n^k)$, $k = 1, ... K$ and probabilities $\alpha_k \geq 0$ with $\Sigma_{k=1}^{K} \alpha_k = 1$, the compound lottery $(L_1, ... , L_K; \alpha_1 , ... , \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for k = 1, ... , K.

2.1.3. *Definition of a reduced lottery.* For any compound lottery we can calculate a corresponding reduced lottery as the simple lottery L = (p_1 , p_2, ... , p_N) that generates the same ultimate distribution over outcomes. The probability of outcome n in the reduced lottery is given by

$$p_n = \alpha_1 p_n^1 + \alpha_2 p_n^2 + \cdots + \alpha_K p_n^K, \quad n = 1, 2, \cdots N \qquad (1)$$

Therefore the reduced lottery of any compound lottery can be obtained by vector addition. Specifically L = α_1 L_1 + ... + α_K L_K $\in \Delta$. Thus compound lotteries lie linearly in the simplex formed from the simple lotteries of which they are formed.

Consider as an example a case with 3 terminal consequences, C = {1,2 3}. Now consider 5 lotteries described as follows:

FIGURE 2. Triangle Representing a Simple Lottery
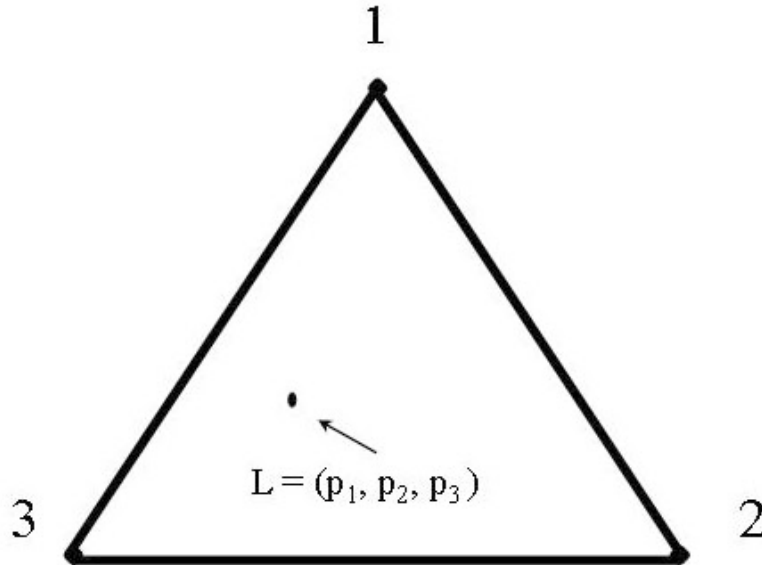
$$L_1 = (1, 0, 0)$$
$$L_2 = (.25, .375, .375)$$
$$L_3 = (.75, .25, 0)$$
$$L_4 = (.5, .125, .375)$$
$$L_5 = (.5, .25, .25)$$

Now consider two compound lotteries. The first gives L_1 with probability .25 and L_5 with probability .75. This leads to a reduced lottery of (.625, .1875, .1875). Consider then the compound lottery that gives L_3 with probability .5 and L_4 with probability .5. This has reduced lottery equal to (.625, .1875, .1875). Thus the two compound lotteries are equivalent.

2.2. **Preferences over lotteries.** We will assume that the set of alternatives to be considered are the set of all simple lotteries over the outcomes C denoted by $\mathcal{L}$. We also assume there exists a binary preference relation on the set of such lotteries.

    1. Continuity or Archimedean axiom

        The preference relation $\succeq$ on the space of simple lotteries $\mathcal{L}$ is continuous if for any (L, L', L'') $\in \mathcal{L}$, the sets

$$\{\alpha \, \epsilon \, [0,1] \, : \, \alpha L \, + \, (1-\alpha)L' \, \succeq \, L"\} \, \subset \, [0,1]$$

$$\{\alpha \, \epsilon \, [0,1] \, : \, L" \, \succeq \, \alpha \, L \, + \, (1-\alpha)L'\} \, \subset \, [0,1] \tag{2}$$

are closed.

As a possible counter example consider the following consequences and simple lotteries.

$$C \, = \, (\$1000, \$10, Death)$$

$$L_1 \, = \, (1,0,0)$$

$$L_2 \, = \, (0,1,0)$$

$$L_3 \, = \, (0,0,1)$$

Assume that $L_1 \succeq L_2 \succeq L_3$. Then there is some compound lottery such that $\alpha \, L_1$ + (1-α) $L_3 \succeq L_2$.

2. Independence axiom

The preference relation $\succeq$ on the space of simple lotteries satisfies the independence axiom if for all (L, L', L") $\in \mathcal{L}$ and $\alpha \in$ (0,1) we have

$$L \succeq L' \iff \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L'' \tag{3}$$

2.3. **The expected utility function.** The utility function U: $\mathcal{L} \to$ R has an expected utility form if there is an assignment of numbers (u_1, u_2, ... , u_N) to the n outcomes such that for every simple lottery L = (p_1, p_2, ... , p_N) $\in \mathcal{L}$, we have

$$U(L) \, = \, u_1 \, p_1 \, + \, u_2 \, p_2 \, + \, \cdots \, + \, u_N \, p_N \, = \, \Sigma_n u_n \, p_n \tag{4}$$

A utility function U: $\mathcal{L} \to$ R with the expected utility form is called a von Neumann-Morgenstern (v.N-M) expected utility function. Note that if the lottery L^n is the lottery that yields outcome n with certainty (p_n =1) then U(L^n) = u_n. The important result is that the utility function is linear in the probabilities.

2.4. **Linearity and expected utility.**

Proposition 1. *A utility function U: $\mathcal{L} \to$ R has an expected utility from iff it is linear, that is iff it satisfies the property that*

$$U \left(\Sigma_{k=1}^K \alpha_k L_k \right) \, = \, \Sigma_{k=1}^K \alpha_k U(L_k) \tag{5}$$

for any K lotteries L_k in $\mathcal{L}$, k = 1, 2, ..., K and probabilities (α_1, ..., α_k) $\geq$ 0, $\Sigma_{k=1}^K \alpha_k = 1$.

Proof. Suppose that U($\cdot$) satisfies equation 5. We can write any lottery L = (p_1, ..., p_N) as a convex combination of the degenerate (certain) lotteries (L^1, ... , L^N), that is $L = \Sigma_n p_n \, L^n$. We then have

$$U(L) = U\left(\Sigma_{n=1}^{N} p_n L^n\right)$$

$$= \Sigma_{n=1}^{N} p_n U(L^n) \tag{6}$$

$$= \Sigma_{n=1}^{N} p_n u_n$$

To show the other way suppose that U has the expected utility form as in equation 4. Now consider any compound lottery $(L_1, \ldots , L_K; \alpha_1, \ldots , \alpha_K)$ where the kth lottery has the form $L_k = (p_1^k, p_2^k, \ldots, p_k^N)$. This compound lottery will have a reduced lottery equivalent to it of the form $L' = \Sigma_k \alpha_k L_k$. Given this we can write the utility of this reduced lottery as

$$U(\Sigma_{k=1}^{K} \alpha_k L_k) = \Sigma_{n=1}^{N} u_n \left(\Sigma_k \alpha_k p_n^k\right)$$

$$= \Sigma_{k=1}^{K} \alpha_k \left(\Sigma_n u_n p_n^k\right) \tag{7}$$

$$= \Sigma_{k=1}^{K} \alpha_k U(L_k)$$

The expected utility property is a cardinal property of utility functions defined on $\mathcal{L}$. This form is preserved by increasing linear transformations as is noted in Proposition 2.

$\square$

Proposition 2. *Suppose that U: $\mathcal{L} \to R$ is a (v.N-M) expected utility function for the preference relation $\succeq$ on $\mathcal{L}$. Then $\tilde{U}$ $\mathcal{L}: \to R$ is another (v.N-M) expected utility function for $\succeq$ iff there are scalars $\beta > 0$ and γ such that $\tilde{U}(L)= \beta U(L)+\gamma$ for every $L \in \mathcal{L}$.*

2.5. **The concentration on monetary outcomes.** We will normally consider monetary outcomes so that more is preferred to less and there are no issues of comparing apples to oranges. In this context the number assigned to an outcome is just some numerical function of money or wealth.

3. EXPECTED UTILITY WHEN THE OUTCOMES ARE CONTINUOUS

3.1. **Notation.**
1. x is a monetary outcome (continuous)
2. F: R $\to$ [0,1] is a cumulative density function (cdf) defining a lottery
3. F(x) is the probability that the realized payoff is less than x
4. f(t) is the density function associated with F if it exists
5. $\mathcal{L}$ is a lottery space which is the set of all distribution functions defined on [a, $+\infty$)

3.2. **Properties of the cumulative density function F.**

$$F(x) = \int_{-\infty}^{x} f(t)\, dt \tag{8}$$

For compound lotteries $(L_1, L_2, \ldots L_K, \alpha_1, \ldots, \alpha_K)$ we have

$$F(x) = \Sigma_{k=1}^{K} \alpha_k F_k(x) \tag{9}$$

3.3. **Expected utility functions.**

3.3.1. *Expected utility with discrete outcomes.*

$$U(L) \ = \ \Sigma_{n=1}^{N} \, p_n u_n$$
$$= p_1 \, u_1 \ + \ p_2 \, u_2 \ + \ \ldots \ p_N \, u_N \tag{10}$$

where u_n is the utility associated with the nth outcome. This is sometimes called the Bernoulli function or preference scaling function.

3.3.2. *Expected utility with continuous outcomes.*

$$U(F) \ = \ \int \ u(x) \, dF(x) \tag{11}$$

where u is the utility associated with the monetary outcome x. As before this is called the Bernoulli or preference scaling function. Often we will write EU(F) for U(F) or if F is dependent on a parameter "a" we will write EU(F(a)) or EU(a) where EU(a) is the expected utility of action a which induces distribution on outcomes denoted by F(a).

3.3.3. *Properties of the function $u(\cdot)$.*

1. increasing
2. continuous
3. bounded (or use restrictions on F)

4. RISK AVERSION

4.1. **Definition of risk aversion in general.** A decision-maker is a risk averter if for any lottery $F(\cdot)$, the degenerate lottery that yields the amount $\int x \, dF(x)$ with certainty $\succeq F(\cdot)$. If the decision maker is always (for any F) indifferent between these two lotteries, we say he is risk neutral. Finally we say that the decision maker is strictly risk neutral if indifference holds only when the two lotteries are the same (F is degenerate).

4.2. **Definition of risk aversion with a v.N-M utility function.** A decision-maker is a risk averter iff

$$\int \ u(x) \, dF(x) \ \leq \ u \left(\int \ x \, dF(x) \right) \ \forall \ F \left(\cdot \right) \tag{12}$$

This is called Jensen's inequality and holds for all concave functions $u(\cdot)$. Strict concavity or strict risk aversion means that the marginal utility of money is decreasing. Thus at any level of wealth the value of a dollar gain is smaller than the utility of the absolute value of the same dollar loss.

4.3. **Example of risk aversion.**

1. States of nature

    Consider two states of nature with $p_1 = p_2 = 0.5$.

2. Preference scaling function

Consider the preference scaling function u( ) = -4 + .17x -.0003x². For this function, the following values are obtained

$$u(100) = 10$$
$$u(150) = 14.75$$
$$u(200) = 18$$
$$u(250) = 19.75$$
$$u(300) = 20$$

3. Lotteries

Consider a lottery where the outcomes are 100 and 300.

4. Expected Utility

U(L) = u(100)(.5) + u(300)(.5) = 10(.5) + 20(.5) = 15.

The expected value of the lottery is E(L) = 100(.5) + 300(.5) = 200. The scaling function implies that u(200) = 18. So U(L) < u(E(L)). An individual who is risk neutral will have a linear utility function u. Consider the shape of the preference scaling function in figure 3. Expected utility is computed along the line connecting the points (100,10) and (300,20). The utility of 200 is higher than the point along this line because u(x) is a concave function.

FIGURE 3. Risk Averse Preference Scaling Function
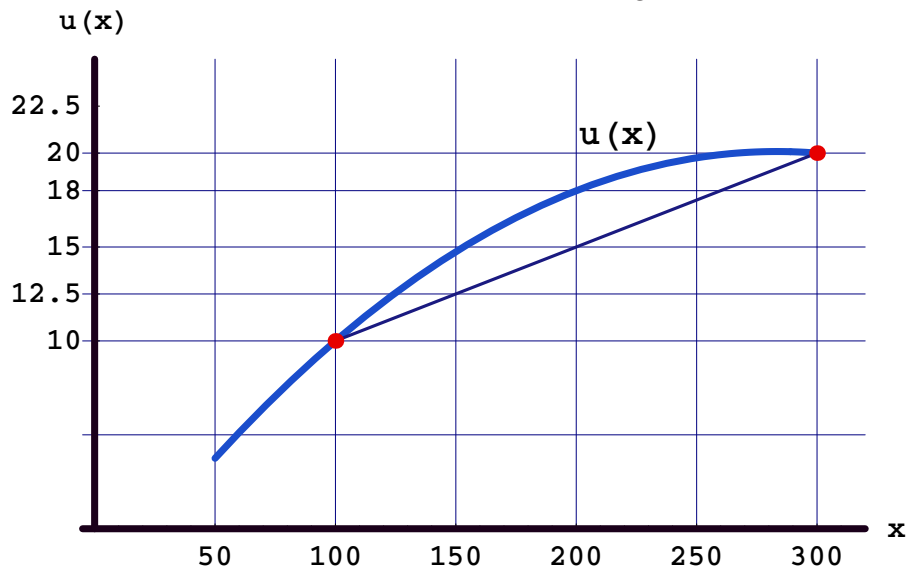
4.4. **Certainty equivalent.** For a given preference scaling function u(·) the certainty equivalent of F(·),denoted c(F,u), is the amount of money for which the individuals is indifferent between the gamble F(·) and the certain amount c(F,u); that is

$$u(c(F, u)) = \int u(x) \, dF(x) \tag{13}$$

For the example we need to find the level of x which has a utility level of 15 which is the expected utility of the lottery giving 100 with probability 0.5 and 300 with probability 0.5. Thus we compute u(?) = 15. Solving the equation for x we obtain

$$
\begin{aligned}
u(x) &= ax^2 + bx + c \\
u_0 &= ax^2 + bx + c \\
&\Rightarrow ax^2 + bx + (c - u_0) = 0 \\
&\Rightarrow (-.0003)x^2 + (.17)x + (-4 - 15) = 0 \\
x &= \frac{-b \pm \sqrt{b^2 - 4a(c - u_0)}}{2a} \\
&= \frac{-.17 \pm \sqrt{(.17)^2 - 4(-.0003)(-19)}}{2(-.0003)} \\
&= \frac{-.17 \pm \sqrt{.0289 - .0228}}{-.0006} \\
&= \frac{-.17 \pm .00781}{-.0006} \\
&\Rightarrow x = 153.162 \; or \; x = 413.51
\end{aligned}
\tag{14}
$$

In figure 4, the certainty equivalent is found by extended a line from the vertical axis at the level of expected utility (15) to the preference scaling function and then reading off the value on the horizontal axis or by extending a line from the point on the expected utility line over to the preference scaling function.

Note that if $c(F,u) \le \int x \, dF(x)$ for all F(·) then the decision maker is a risk averter. This can be seen by noting that u(·) is nondecreasing and writing the following expressions starting with the assumption that the certainty equivalent is less than the expected return

$$
\begin{aligned}
c(F, u) &\le \int x \, dF(x), \quad \text{assumption} \\
\Leftrightarrow u(c(F, u)) &\le u \left(\int x \, dF(x) \right), \quad \text{u nondecreasing} \\
\Leftrightarrow \int u(x) \, dF(x) &\le u \left(\int x \, dF(x) \right) \text{definition of c(F,u)}
\end{aligned}
\tag{15}
$$

FIGURE 4. Finding the Certainty Equivalent
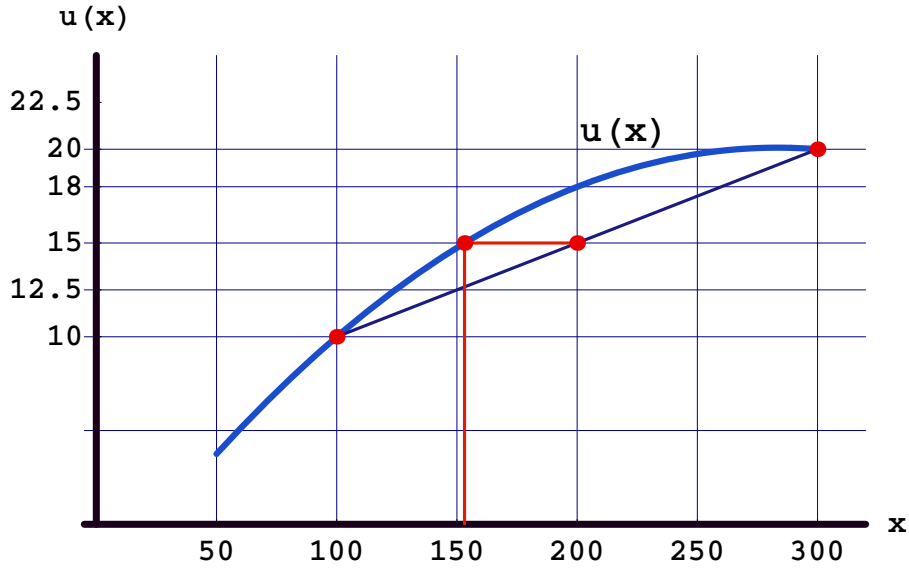
4.5. **probability premium.** For any fixed amount of money x and a positive number ϵ, the probability premium demoted by $\pi(x,\epsilon,u)$, is the excess in wining probability over fair odds that makes the individual indifferent between the certain outcome x and a gamble between the two outcomes x+ϵ and x-ϵ. That is

$$u(x) \; = \; \left(\frac{1}{2} \, + \, \pi(x, \epsilon, u) \right) \, u(x + \epsilon) \, + \, \left(\frac{1}{2} \, - \, \pi(x, \epsilon, u) \right) \, u(x - \epsilon) \qquad (16)$$

For any given x and ϵ we can compute π as follows:

$$
\begin{aligned}
u(x) \; &= \; \left(\frac{1}{2} \, + \, \pi(x, \epsilon, u) \right) \, u(x + \epsilon) \, + \, \left(\frac{1}{2} \, - \, \pi(x, \epsilon, u) \right) \, u(x - \epsilon) \\[2mm]
&= \frac{1}{2} \, [\, u(x + \epsilon) \, + \, u(x - \epsilon) \,] \, + \, \pi \, [\, u(x + \epsilon) \, - \, u(x - \epsilon) \,] \\[2mm]
\Rightarrow \; &u(x) \, - \, \frac{1}{2} \, [\, u(x + \epsilon) \, + \, u(x - \epsilon) \,] \; = \; \pi \, [\, u(x + \epsilon) \, - \, u(x - \epsilon) \,] \\[2mm]
\Rightarrow \; &\pi \; = \; \frac{u(x) \, - \, \frac{1}{2} \, [\, u(x + \epsilon) \, + \, u(x - \epsilon) \,]}{[\, u(x + \epsilon) \, - \, u(x - \epsilon) \,]}
\end{aligned}
\qquad (17)
$$

For the example given we can compute the probability premium needed to make the decision maker indifferent between a certain outcome of 200 [u(200) = 18] and a gamble between 100 and 300 with respective utilities of 10 and 20. In this case, ϵ = 100. This will give

$$\begin{aligned}
\pi &= \frac{u(x) - \frac{1}{2}\left[u(x+\epsilon) + u(x-\epsilon)\right]}{\left[u(x+\epsilon) - u(x-\epsilon)\right]} \\
&= \frac{u(200) - \frac{1}{2}\left[u(300) + u(100)\right]}{\left[u(300) - u(100)\right]} \\
&= \frac{18 - \frac{1}{2}\left[20 + 10\right]}{\left[20 - 10\right]} \\
&= \frac{18 - 15}{\left[10\right]} = \frac{3}{10}
\end{aligned} \tag{18}$$

Checking we obtain

$$\begin{aligned}
u(x) &= \left(\frac{1}{2} + \pi(x,\epsilon,u)\right) u(x+\epsilon) + \left(\frac{1}{2} - \pi(x,\epsilon,u)\right) u(x-\epsilon) \\
u(200) &= \left(\frac{1}{2} + \frac{3}{10}\right) u(300) + \left(\frac{1}{2} - \frac{3}{10}\right) u(100) \\
18 &= \left(\frac{8}{10}\right)(20) + \left(\frac{2}{10}\right)(10) \\
18 &= (16 + 2) = 18
\end{aligned} \tag{19}$$

We can examine this graphically in figure 5. Here, u(200) = 18, u(200-ϵ) = u(100) = 10, and u(200+ϵ) = u(300) = 20. The line for the vertical axis at 18 over to the expected utility line for the lottery for different probabilities for 200-ϵ and 200+ϵ indicates that the probability must be more than one-half of the distance between these two outcomes. The vertical line indicates that a lottery between 100 and 300 with an expected wealth level of 260 has a utility level of 18.

If the decision maker is risk neutral then u(x) = x and the probability premium is given by

$$\begin{aligned}
u(x) &= \left(\frac{1}{2} + \pi(x,\epsilon,u)\right) u(x+\epsilon) + \left(\frac{1}{2} - \pi(x,\epsilon,u)\right) u(x-\epsilon) \\
\Rightarrow x &= \left(\frac{1}{2} + \pi(x,\epsilon,u)\right)(x+\epsilon) + \left(\frac{1}{2} - \pi(x,\epsilon,u)\right)(x-\epsilon) \\
\Rightarrow x &= x + \left(\frac{1}{2} + \pi(x,\epsilon,u)\right)(x+\epsilon) + \left(\frac{1}{2} - \pi(x,\epsilon,u)\right)(x-\epsilon) \\
\Rightarrow x &= x + 2\,\pi(x,\epsilon,u)\,\epsilon \\
\Rightarrow 0 &= 2\,\pi(x,\epsilon,u)\,\epsilon \\
&\Rightarrow \pi(x,\epsilon,u) = 0 \; if \; \epsilon \neq 0
\end{aligned} \tag{20}$$

Now consider the utility function given by the straight line through the points (100,10) and (300,20). This can be determined as follows where u(x*) is a fixed number based on the chosen value of x*.

FIGURE 5. Finding the Probability Premium
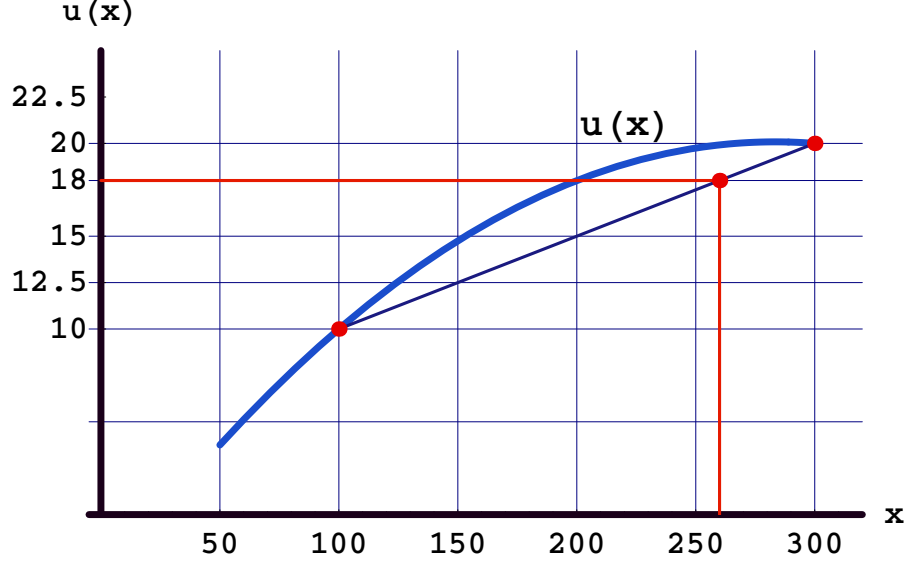
$$20 - u(x^*) = \frac{20 - 10}{(300) - (100)} ((300) - x)$$

$$[4pt]20 - u(x^*) = \frac{10}{200} (300 - x)$$

$$5 - u(x^*) = \frac{-1}{20} x \tag{21}$$

$$u(x^*) = \frac{1}{20}x + 5$$

$$\Rightarrow x = 20 (u(x^*) - 5)$$

Equation 21 gives the value of x that will lead to the same utility as the target level along the straight line representing a lottery between outcomes $x^* + \epsilon$ and $x^* - \epsilon$. In general this is given by

$$u(x^* + \epsilon) - u(x^*) = \frac{u(x^* + \epsilon) - u(x^* - \epsilon)}{(x^* + \epsilon) - (x^* - \epsilon)} ((x^* + \epsilon) - x)$$

$$u(x^* + \epsilon) - u(x^*) = \frac{u(x^* + \epsilon) - u(x^* - \epsilon)}{2\epsilon} ((x^* + \epsilon) - x)$$

$$u(x^*) = u(x^* + \epsilon) - \frac{u(x^* + \epsilon) - u(x^* - \epsilon)}{2\epsilon} ((x^* + \epsilon) - x) \tag{22}$$

$$\Rightarrow x = (x^* + \epsilon) - \frac{2\epsilon(u(x^* + \epsilon) - u(x^*))}{u(x^* + \epsilon) - u(x^* - \epsilon)}$$

We can compute π in a slightly different way by setting u(200) = 18 equal to average of the upper and lower utilities

$$u(x) = \left(\frac{1}{2} + \pi(x, \epsilon, u) \right) u(x + \epsilon) + \left(\frac{1}{2} - \pi(x, \epsilon, u) \right) u(x - \epsilon)$$

$$u(200) = 18 = \left(\frac{1}{2} + \pi(x, \epsilon, u) \right) u(300) + \left(\frac{1}{2} - \pi(x, \epsilon, u) \right) u(100)$$

$$18 = \frac{1}{2} \left(u(300) + u(100) \right) + \pi(x, \epsilon, u) \left(u(300) - u(100) \right) \tag{23}$$

$$18 = \frac{1}{2} (30) + \pi(x, \epsilon, u) (10)$$

$$3 = \pi(x, \epsilon, u) (10)$$

$$\Rightarrow \pi(x, \epsilon, u) = \frac{3}{10}$$

The point on the x axis associated with this probability level is

$$u(x^*) = \frac{1}{20} x + 5$$

$$x = 20 \left(u(x^*) - 5 \right) \tag{24}$$

$$x = 20(18 - 5)$$

$$= 260$$

4.6. **Equivalent characterizations of risk aversion.** Suppose the decision maker is an expected utility maximizer with a Bernoulli utility (preference scaling) function $u(\cdot)$ on amounts of money. Then the following are equivalent:

1. The decision maker is risk averse
2. $u(\cdot)$ is concave ($u''(x) \leq$)
3. $c(F,u) \leq \int x dF(x)$ for all $F(\cdot)$
4. $\pi(x,\epsilon,u) \geq 0$ for all x, ϵ

4.7. **Risk Aversion Example.** Suppose an investor can choose between two assets. Asset one has a random return of z per unit invested and asset two has a certain return of x per unit invested. Assume that the investor allocates α dollars to the first asset and β dollars to the second asset where $\alpha + \beta$ = wealth (w). Given any particular random return the portfolio pays $\alpha z + \beta x$. The utility maximization problem can be written as follows

$$\max_{\alpha, \beta \geq 0} \int u(\alpha z + \beta x) \, dF(z) \tag{25}$$

$$s.t. \ \alpha + \beta = w$$

If we substitute for β from the constraint we obtain

$$\max \int u(wx + \alpha(z - x))\, dF(z)$$

$$s.t.\ 0 \le \alpha \le w\ or$$

$$\max \int u(wx + \alpha(z - x))\, dF(z) \tag{26}$$

$$s.t.\ \alpha \ge 0$$

$$(w - \alpha) \ge 0$$

This is a nonlinear programming problem with two constraints on the decision variable α. The associated Lagrangian is

$$\mathcal{L} \ = \ \int u(wx + \alpha(z - x))\, dF(z) + \lambda_1 \alpha \ + \ \lambda_2 (w - \alpha) \tag{27}$$

The first order conditions are

$$\int u'(wx \ + \ \alpha(z - x))\,(z - x)\, dF(z) + \lambda_1 \ - \ \lambda_2 \ = \ 0$$

$$\lambda_1\, \alpha \ = \ 0 \tag{28}$$

$$\lambda_2\,(w - \alpha) \ = \ 0$$

$$\lambda_1,\ \lambda_2 \ \ge 0$$

If $\alpha > 0$ then $\lambda_1 = 0$ and we have that

$$\int u'(wx + \alpha(z - x))(z - x)\, dF(z) \ = \ \lambda_2 \ \ge \ 0 \tag{29}$$

because $\lambda_2 \ge 0$. If $\alpha < $ w then $\lambda_2 = 0$ and we have that

$$\int u'(wx + \alpha(z - x))(z - x)\, dF(z) \ = \ -\lambda_1 \ \le \ 0 \tag{30}$$

For this function to be a maximum we need to check the second order conditions . If the objective function is concave and the constraints are also concave this stationary point will be a maximum. The objective function is concave because u is concave. This is obvious from differentiation

$$\int u''(wx + \alpha(z - x))(z - x)^2\, dF(z) \ \le \ 0 \tag{31}$$

The constraints are linear and therefore concave.

Now consider the case if the risky asset has an expected return greater than x. That is $\int z\, dF(z) > x$. Now consider the possibility of $\alpha = 0$ as the solution to this problem. If $\alpha \ = \ 0$ we obtain

$$\int u'\,(wx)(z-x)\,dF(z) \,+\, \lambda_1 \,-\, \lambda_2 \,=\, 0$$

$$\lambda_1\,\alpha \,=\, 0$$

$$\lambda_2\,(w-\alpha) \,=\, 0 \tag{32}$$

$$\lambda_1,\ \lambda_2 \,\geq\, 0$$

We can rewrite the integral in equation 32 as follows

$$\int u'\,(wx)(z-x)\,dF(z) \,+\, \lambda_1 \,-\, \lambda_2 \,=\, u'\,(wx)\,\left(\int z\,dF(z) \,-\, x\int dF(z)\right) \,+\, \lambda_1 \,-\, \lambda_2$$

$$=\, u'\,(wx)\,\left(\int z\,dF(z) \,-\, x\right) \,+\, \lambda_1 \,-\, \lambda_2 \tag{33}$$

Because the expression in brackets is positive by the assumption we have that a positive number $\left[(u'(wx)\left(\int z\,dF(z) \,-\, x\int dF(z)\,\right)\right] \,+\, \lambda_1 \,-\, \lambda_2$ is equal to zero. Because both λ's are positive this implies that $\left[(u'(wx)\left(\int z\,dF(z) \,-\, x\int dF(z)\,\right)\right]$ = $-\lambda_1 + \lambda_2$. But if $\alpha < w$ then $\lambda_2 = 0$. Because $\lambda_1 \geq 0$ we have a contradiction. Thus $\alpha \neq 0$. So a risk averse investor will always invest some money in the risky asset if it has a higher return than the safe asset.
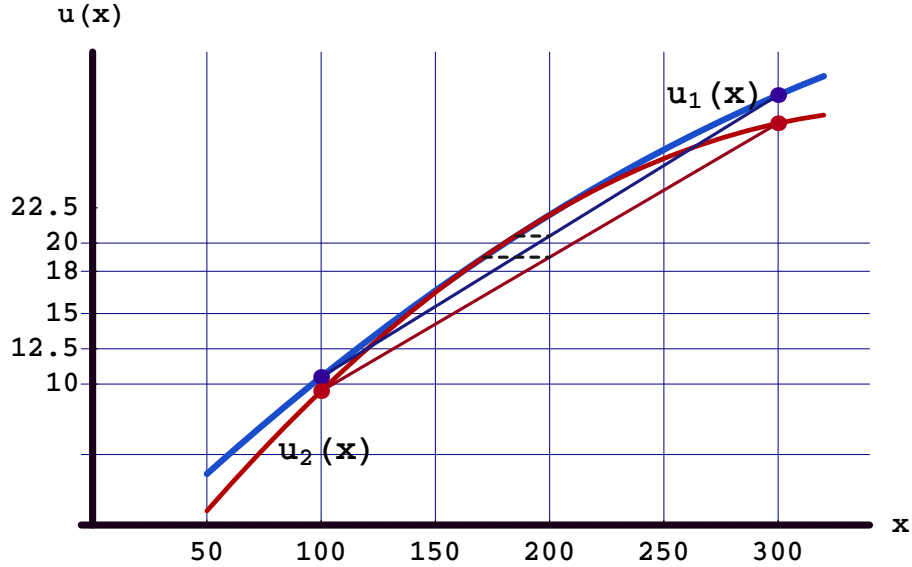
5. MEASUREMENT OF RISK AVERSION

5.1. **Arrow-Pratt coefficient of absolute risk aversion.** Given a twice differentiable preference scaling function u(·) for money, the Arrow-Pratt coefficient of absolute risk aversion at the point x is defined as

$$r_A(x) \; = \; -\frac{u''(x)}{u'(x)} \tag{34}$$

With risk neutrality, u is linear and u″ = 0. Thus r_A measures the curvature of the preference scaling function. The use of u′ in the denominator makes it invariant to positive linear transformations. Consider figure 6 where $u_1(\cdot)$ is less curved than $u_2(\cdot)$. It is obvious that the certainty equivalent is less for the more curved function.

FIGURE 6. Finding the Certainty Equivalent

The coefficient of risk aversion can also be related to the probability premium by differentiating the defining identity (equation 16 twice with respect to ϵ and then evaluating at ϵ = 0. Taking the first derivative will give

$$u(x) \; = \; \left(\frac{1}{2} + \pi(x,\epsilon,u)\right) u(x+\epsilon) \; + \; \left(\frac{1}{2} - \pi(x,\epsilon,u)\right) u(x-\epsilon)$$

$$0 = \left(\frac{d\pi(x,\epsilon,u)}{d\epsilon}\right) u(x+\epsilon) + \left(\frac{1}{2} + \pi(x,\epsilon,u)\right) u'(x+\epsilon) + \left(\frac{-d\pi(x,\epsilon,u)}{d\epsilon}\right) u(x-\epsilon) - \left(\frac{1}{2} - \pi(x,\epsilon,u)\right) u'(x-\epsilon)$$

$$= \pi'u(x+\epsilon) + \frac{1}{2} u'(x+\epsilon) + \pi u'(x+\epsilon) - \pi'u(x-\epsilon) - \frac{1}{2} u'(x-\epsilon) + \pi u'(x-\epsilon) \tag{35}$$

Differentiating again will give

$$
\begin{aligned}
0 &= \pi'u(x+\epsilon) + \frac{1}{2}\,u'(x+\epsilon) + \pi u'(x+\epsilon) - \pi'u(x-\epsilon) - \frac{1}{2}\,u'(x-\epsilon) + \pi u'(x-\epsilon) \\
&= \pi''u(x+\epsilon) + \pi'u'(x+\epsilon) + \frac{1}{2}\,u''(x+\epsilon) + \pi'u'(x+\epsilon) + \pi u''(x+\epsilon) \\
&\quad - \pi''u(x-\epsilon) + \pi'u'(x-\epsilon) + \frac{1}{2}\,u''(x-\epsilon) + \pi'u'(x-\epsilon) - \pi u''(x-\epsilon)
\end{aligned}
\tag{36}
$$

Now evaluate at $\epsilon = 0$ to obtain

$$
\begin{aligned}
0 &= \pi''u(x) + \pi'u'(x) + \frac{1}{2}\,u''(x) + \pi'u'(x) + \pi u''(x) \\
&\quad - \pi''u(x) + \pi'u'(x) + \frac{1}{2}\,u''(x) + \pi'u'(x) - \pi u''(x) \\
&= 4\pi'u'(x) + u''(x) \\
\Rightarrow \frac{-u''(x)}{u'(x)} &= 4\pi'(0)
\end{aligned}
\tag{37}
$$

Continuing will give $r_A(x) = 4\pi'(0)$. Note that the utility function can be obtained from $r_A(\cdot)$ by integrating twice. The two constants are irrelevant since the Bernoulli utility function is only identified up to linear transformations.

5.2. **Example with Constant Absolute Risk Aversion (CARA).** Let the preference scaling function be given by $u(x) = -e^{-kx}$, $k > 0$. This is known as the negative exponential utility function. For this function, $u'(x) = ke^{-kx}$ and $u'(x) = -k^2 e^{kx}$ and $r_A(x,u) = k$ for all x. Similarly we can obtain for $r_A(x) = k$ that

$$
\begin{aligned}
r_A(x) &= -\frac{u''(x)}{u'(x)} \\
\Rightarrow k &= -\frac{u''(x)}{u'(x)} \\
\Rightarrow \frac{d(\log u'(x))}{dx} &= -k \\
\Rightarrow \log u'(x) &= -kx + \ln c \\
\Rightarrow u'(x) &= e^{-kx + \log c} = e^{-kx}e^{\log c} = ce^{-kx} \\
\Rightarrow u(x) &= \frac{-c}{k}e^{-kx} + b \\
&= -ae^{-kx} + b
\end{aligned}
\tag{38}
$$

5.3. **Relative risk aversion.** The coefficient of relative risk aversion for a given Bernoulli utility function is given by

$$
r_R(x,u) = -\frac{x\,u''(x)}{u'(x)}
\tag{39}
$$

5.3.1. *Log utility functions.* The log utility function is given by

$$u(x) \; = \; \beta \log x \; + \; \gamma, \quad \beta > 0 \tag{40}$$

The coefficient of relative risk aversion for the log utility function is obtained by differentiating equation 40 with respect to x.

$$u'(x) = \frac{\beta}{x}$$

$$u''(x) \; = \; -\frac{\beta}{x^2}$$

$$\Rightarrow \; r_R(x) \; = \; \frac{-x \left(-\frac{\beta}{x^2} \right)}{\left(\frac{\beta}{x} \right)} \; = \; 1 \tag{41}$$

5.3.2. *Power utility functions.* The power utility function is given by

$$u(x) \; = \; \beta x^{1-\rho} \; + \; \gamma, \quad \beta > 0, \, \rho \neq 1 \tag{42}$$

The power utility function collapses to the log utility function as $\rho \to 1$. The coefficient of relative risk aversion for the power utility function is obtained by differentiating equation 42 with respect to x.

$$u'(x) \; = (1 - \rho) \, \beta x^{-\rho}$$

$$u''(x) \; = (-\rho)(1 - \rho)\beta x^{-\rho-1}$$

$$\Rightarrow \; r_R(x) \; = \frac{(-x)\,(-\rho)(1 - \rho)\beta x^{-\rho-1}}{(1 - \rho)\,\beta x^{-\rho}} \; = \; \rho \tag{43}$$

Notice that the coefficient of relative risk aversion is constant for both of these utility functions, i.e., it does not depend on wealth.

6. RISK AVERSION AS A FUNCTION OF WEALTH

6.1. **Definition of Decreasing Absolute Risk Aversion (DARA).** The preference scaling function u(·) exhibits decreasing absolute risk (DARA) aversion if $r_A(x)$ is a decreasing function of x. Individuals with DARA take more risk as they become wealthier.

Proposition 3. *The following properties are equivalent:*

1. *The Bernoulli utility or preference scaling function exhibits decreasing absolute risk aversion.*
2. *Whenever $x_2 < x_1$, $u_2(z) = u(x_2+z)$ is a concave transformation of $u_1(z) = u(x_1 + z)$.*
3. *For any risk F(z), the certainty equivalent of the lottery formed by adding risk z to wealth level x, given by the amount c_x at which $u(c_x) = \int u(x+z)dF(z)$, is such that $(x-c_x)$ is decreasing in x. That is, the higher x is, the less the individual is willing to pay to get rid of the risk.*
4. *The probability premium $\pi(x,\epsilon,u)$ is decreasing in x.*

6.2. **Definition of Decreasing Relative Risk Aversion (DRRA).** The preference scaling function u($\cdot$) exhibits decreasing relative risk(DRRA) aversion if $r_{\mathrm{R}}(x)$ is a decreasing function of x. Individuals with DRRA become less risk averse with respect to gambles that are proportional to wealth as wealth increases. A person with decreasing relative risk aversion will also exhibit decreasing absolute risk aversion. The converse is not necessarily true.

Proposition 4. *The following properties are equivalent:*

1. *The Bernoulli utility function exhibits decreasing relative risk aversion, i.e. $r_{\mathrm{R}}(x,u)$ is decreasing in x.*
2. *Whenever $x_2 < x_1$, $\tilde{u}_2(t) = u(tx_2)$ is a concave transformation of $\tilde{u}_1(t) = u(tx_1)$.*
3. *Given any risk F(t) on $t > 0$, the certainty equivalent $\bar{c}_x$ defined by $u(\bar{c}_x) = \int u(tx)\,dF(t)$ is such that $\frac{x}{\bar{c}_x}$ is decreasing in x.*