CONSUMER CHOICE AND DUALITY

1. Duality Relationships

1.1. Utility Function. The utility maximization problem for the consumer is as follows

$$\max_{x \geq 0} v(x)$$
$$s.t.\ px \ \leq \ m \tag{1}$$

where we assume that p $>>$ 0, m $>$ 0 and X = R_+^L . The solution to 1 is given by x(p,m) = g(p,m). These functions are called Marshallian demand equations. Note that they depend on the prices of all good and income. This is called the *primal* preference problem. If we substitute the optimal values of the decision variables x into the utility function we obtain the indirect utility function. For the utility maximization problem this gives

$$u \ = \ v\,(x_1,\ x_2,\ \cdots,\ x_n\,) \ = \ v\,[x_1\,(m,\ p),\ x_2\,(m,\ p),\ \ldots,\ x_n\,(m,\ p)\,] \ = \ \psi(m,\,p) \tag{2}$$

The indirect utility function specifies utility as a function of prices and income. We can also write it as follows

$$\psi\,(m,\ p) \ = \ \max_x\,[v(x) :\ px \ = \ m] \tag{3}$$

Given that the indirect utility function is homogeneous of degree zero in prices and income, it is often useful to write it in the following useful fashion.

$$\begin{aligned} \psi\,(m,\ p) \ &= \ \max_x\,[v(x) :\ px \ = \ m] \\ &= \ \max_x\,[v(x) :\ \left(\frac{p}{m}\right) x \ = \ 1] \\ &= \ \max_x\,[v(x) :\ qx \ = \ 1], \quad q \ = \ \frac{p}{m} \ = \ \left\{\frac{p_1}{m}, \frac{p_2}{m}, \ldots, \frac{p_n}{m}\right\} \\ &= \psi\,(q) \end{aligned} \tag{4}$$

We can obtain the utility function from the indirect utility function as follows.

$$u(x) \ = \min_{q \geq 0} \quad \psi(q)$$
$$s.t.\ qx \ \leq \ 1 \tag{5}$$

We can obtain the utility function from the cost function as follows.

$$u(x) \ = \max\ u$$
$$s.t.\ c(u,p) \ \leq \ px,\ \forall p \in R_{++}^n \tag{6}$$

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1.2. **The expenditure (cost) minimization problem.** The fundamental (primal) utility maximization problem is given by

$$\begin{aligned} \max_{x \geq 0} u &= v(x) \\ s.t.\ px &\leq m \end{aligned} \tag{7}$$

Dual to the utility maximization problem is the cost minimization problem

$$\begin{aligned} \min_{x \geq 0} m &= px \\ s.t.\ v(x) &= u \end{aligned} \tag{8}$$

The solution to equation 8 gives the Hicksian demand functions x = h(u, p). The Hicksian demand equations are sometimes called ”compensated” demand equations because they hold u constant. The solutions to the primal and dual problems coincide in the sense that

$$x = g(p, m) = h(u, p) \tag{9}$$

For the dual problem the indirect objective function is

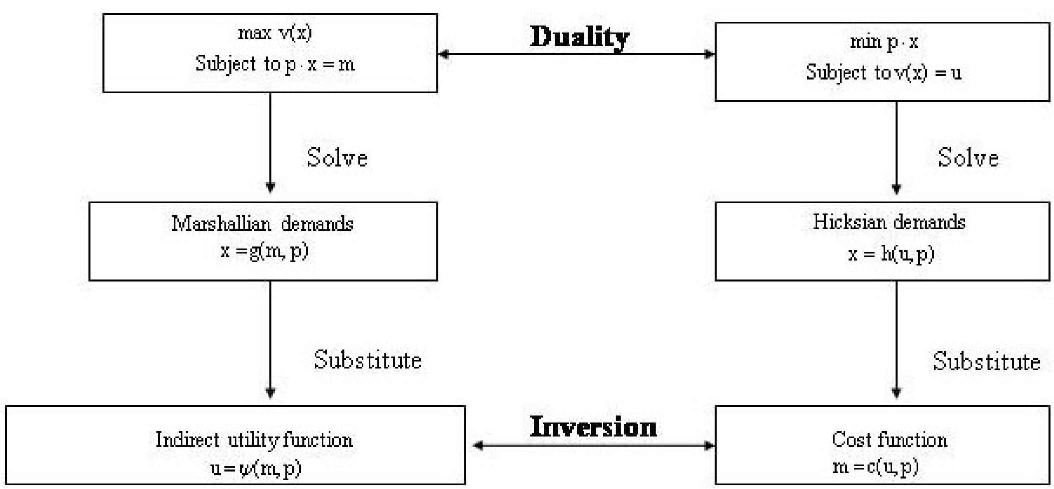
$$m = \Sigma_{j=1}^{n} p_j\, h_j u, p) = c(u, p) \tag{10}$$

This is called the cost (expenditure) function and specifies cost or expenditure as a function of prices and utility. We can also write it as follows

$$c(u, p) = \min_{x} [p\, x : v(x) = u] \tag{11}$$

Because c(u, p) = m, we can rearrange or invert it to obtain u as a function of m and p. This will give ψ(m, p). Similarly inversion of ψ(m,p) will give c(u, p). These relationships between the utility maximization cost minimization problems are summarized in figure 1

FIGURE 1. Utility Maximization and Cost Minimization



1.2.1. *Shephard's Lemma.* If indifference curves are convex, the cost minimizing point is unique. Then we have

$$\frac{\partial C(u,\ p)}{\partial p_i} \ = \ h_i\,(u,\ p) \tag{12}$$

which is a Hicksian Demand Curve. If we substitute the indirect utility function in the Hicksian demand functions obtained via Shephard's lemma in equation 12, we get x in terms of m and p. Specifically

$$x_i \ = \ x_i(u,\ p) \ = \ h_i(u,\ p) \ = \ h_i[\psi(m,\ p),\ p] \ = \ g_i(m,\ p) \ = \ x_i(m,\ p) \tag{13}$$

1.3. **The indirect utility function and Hicksian demands.** If we substitute C(u,p) in the Marshallian demands, we get the Hicksian demand functions

$$x_i \ = \ x_i(m,\ p) \ = \ g_i(m,\ p) \ = \ g_i[C(u,p),p] \ = \ h_i(u,p) \ = \ x_i(u,\ p) \tag{14}$$

1.4. **Roy's identity.** We can also rewrite Shephard's lemma in a different way. First write the identity

$$\psi\,(C(u,p),p] \ = \ u \tag{15}$$

Then totally differentiate both sides of equation 15with respect to p_i holding u constant as follows

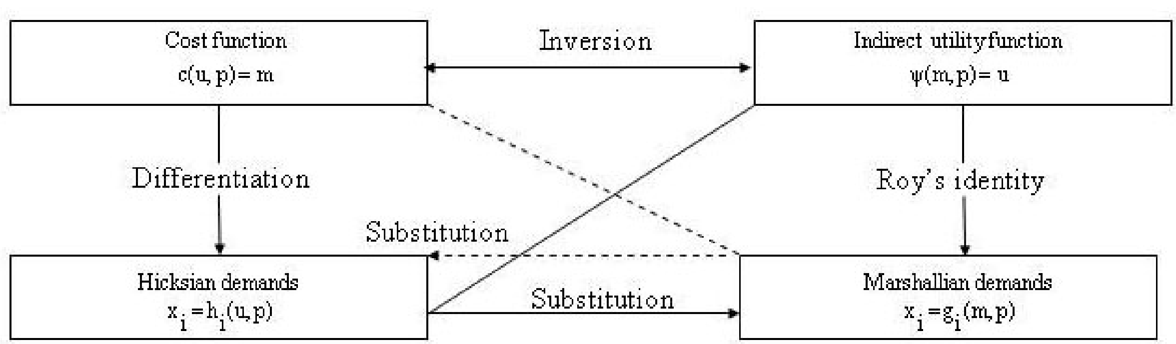
$$\frac{\partial\,\psi[C(u,p),\ p]}{\partial m}\,\frac{\partial\,C(u,p)}{\partial\,p_i} + \frac{\partial\,\psi[C(u,p),p]}{\partial\,p_i} \ = \ 0 \tag{16}$$

Rearranging we obtain

$$\frac{\partial\,C(u,p)}{\partial\,p_i} \ = \ \frac{-\frac{\partial\,\psi\,[C(u,p),p]}{\partial\,p_i}}{\frac{\partial\,\psi\,[C(u,p),p]}{\partial\,m}} \ = \ g_i(m,p) \tag{17}$$

where the last equality follows because we are evaluating the indirect utility function at income level m. Figure 2 makes these relationships clear.

FIGURE 2. Demand, Cost and Indirect Utility Functions



1.5. **Money Metric Utility Functions.** Assume that the consumption set X is closed, convex, and bounded from below. The common assumption that the consumption set is $X = R_+^L = \{x \in R^L\colon x_\ell \geq 0 \text{ for } \ell = 1, \ldots, L\}$ is more than sufficient for this purpose. Assume that the preference ordering satisfies the normal properties. Then for all $x \in X$, let $BT(x) = \{y \in BT \mid y \succeq x\}$. For the price vector p, the money metric m(p,x) is defined by

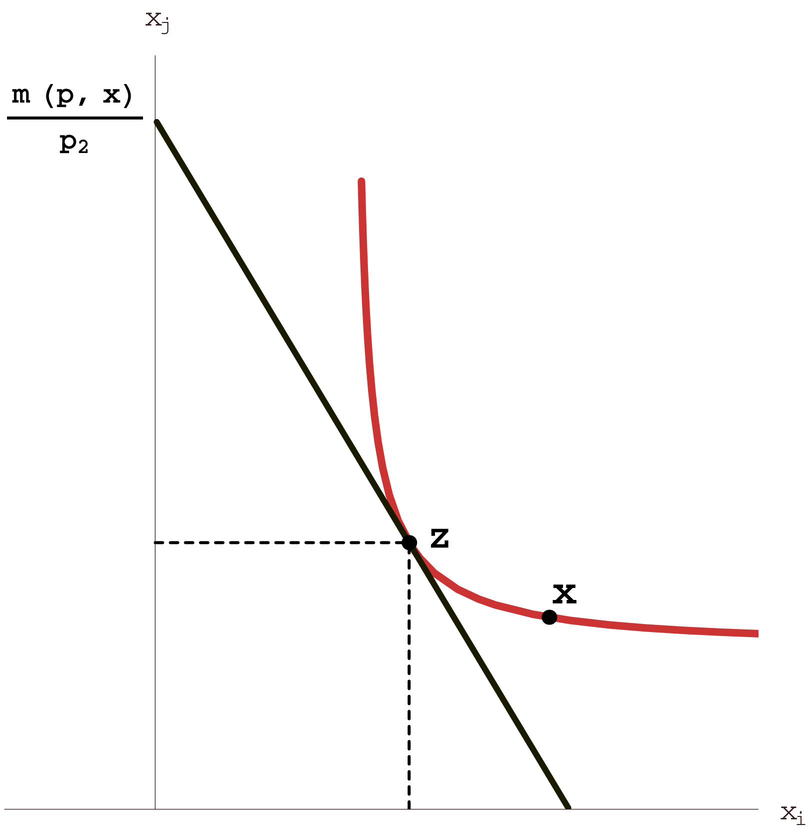
$$\begin{aligned} m(p, x) \; &= \; \min_{y \geq 0} \; py \\ s.t. \; y \; &\in \; BT(x) \end{aligned} \tag{18}$$

If p is strictly greater than zero and if x is a unique element of the least cost commodity bundles at prices p, then m(p,x) can be viewed as a utility function for a fixed set of prices. It can also be defined as follows.

$$m(p, x) \; = \; C(u(x), p) \tag{19}$$

The money metric defines the minimum cost of buying bundles as least as good as x. Consider figure 3

FIGURE 3. Utility Maximization and Cost Minimization



All points on the indifference curve passing through x will be assigned the same level of m(p,x), and all points on higher indifference curves will be assigned a higher level.