MERGERS

1. Types of mergers

1.1. Horizontal. Horizontal mergers are mergers between firms which were formerly competitors. A horizontal merger involves two firms who produce products that are considered substitutes by their buyers or by two firms who purchase the same or substitute products in the input market.

1.2. Vertical. Vertical mergers are mergers between firms at different stages of the production chain or more generally between firms that produce complementary goods such as feeder cattle and slaughter cattle, crude oil and gasoline, broiler production and broiler processing, chemical herbicides and particular types of seed, communications firms operating in different markets, etc.

1.3. Conglomerate. Conglomerate mergers are mergers between firms without a clear substitute or clear complementary relationship.

2. Horizontal mergers

2.1. A Simple Cournot Model.

2.1.1. *Assumptions and structure of the model.*

1. $N \geq 2$ firms in the industry
2. Identical and constant costs for each firm given by $C(q_j) = cq_j$, j = 1,2, ... , N
3. Linear inverse demand given by

$$\begin{aligned} p &= A - BQ \\ &= A - B\,(q_1 + \cdots + q_{i-1} + q_i + q_{i+1} + \cdots + q_N) \\ &= A - B\,(\Sigma_{j \neq i}\, q_j) - B\, q_i \\ &= A - B\, Q_{-i} - B\, q_i \end{aligned} \tag{1}$$

4. Industry supply given by $Q = Q_{-i} + q_i$, $Q_{-i} = \Sigma_{j \neq i}\, q_j$
5. Profit for the i^{th} firm given by

$$\begin{aligned} \pi_i &= \max_{q_i} \,[p\, q_i - C(q_i)] \\ &= \max_{q_i} \,[(A - B\, Q_{-i} - B\, q_i)\, q_i - c\, q_i] \\ &= \max_{q_i} \,[A\, q_i - B\, Q_{-i}\, q_i - B\, q_i^2 - c\, q_i] \end{aligned} \tag{2}$$

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2.1.2. *Profit maximization for a representative firm.* Optimizing the profit expression with respect to q_i will give

$$\begin{aligned} \pi_i &= A\,q_i - B\,Q_{-i}\,q_i - B\,q_i^2 - c\,q_i \\ \frac{d\pi_i}{dq_i} &= A - B\,Q_{-i} - 2\,B\,q_i - c = 0 \\ \Rightarrow\ 2\,B\,q_i &= A - B\,Q_{-i} - c \\ \Rightarrow\ q_i^* &= \frac{A - c}{2\,B} - \frac{Q_{-i}}{2} \end{aligned} \tag{3}$$

Since all firms are identical, this holds for all other firms too.

2.1.3. *Nash equilibrium.* In a Nash equilibrium, each firm i chooses a best response q_i^* that reflects a correct prediction of the other N-1 outputs in total. Denote this sum as Q_{-i}^*. A Nash equilibrium is then

$$q_i^* = \frac{A - c}{2\,B} - \frac{Q_{-i}^*}{2} \tag{4}$$

Because all the firms are identical we can write $Q_{-i}^* = \Sigma_{i \neq j}\, q_j^* = (N-1)\,q^*$ where q* is the optimal output for the identical firms. We can then write

$$\begin{aligned} q^* &= \frac{A - c}{2\,B} - \frac{(N-1)\,q^*}{2} \\ \Rightarrow\ \frac{2\,q^* + (N-1)q^*}{2} &= \frac{A - c}{2\,B} \\ \Rightarrow\ \frac{(N+1)q^*}{2} &= \frac{A - c}{2\,B} \\ \Rightarrow\ q^* &= \frac{A - c}{(N+1)\,B} \end{aligned} \tag{5}$$

Industry output and price are given by

$$\begin{aligned} Q^* &= \frac{N\,(A - c)}{(N+1)\,B} \\ \\ p^* &= A - BQ^* \\ &= A - B\left(\frac{N\,(A - c)}{(N+1)\,B}\right) \\ &= \frac{(N+1)\,A}{N+1} - \frac{N\,A}{(N+1)} - \frac{N\,c}{N+1} \\ &= \frac{A}{N+1} + \frac{N}{N+1}\,c \end{aligned} \tag{6}$$

The i^{th} firm has profit equal to

$$
\begin{aligned}
\pi_i &= p^* q_i^* - c\, q_i^* \\
&= \left(\frac{A}{N+1} + \frac{N}{N+1}\, c \right) q_i^* - c\, q_i^* \\
&= \left(\frac{A}{N+1} + \left(\frac{N}{N+1} - 1 \right) c \right) q_i^* \\
&= \left[\frac{A}{N+1} + \left(\frac{N}{N+1} - 1 \right) c \right] \left[\frac{A-c}{(N+1)\,B} \right] \\
&= \left[\frac{A}{N+1} + \left(\frac{N - N - 1}{N+1} \right) c \right] \left[\frac{A-c}{(N+1)\,B} \right] \\
&= \left[\frac{A-c}{N+1} \right] \left[\frac{A-c}{(N+1)\,B} \right] \\
&= \frac{(A-c)^2}{(N+1)^2 B}
\end{aligned} \tag{7}
$$

2.1.4. *Merger of $M \leq N$ firms.* If M of the N firms in the industry merge then there are N-M+1 independent firms left in the industry. This merged firm is comprised of firms 1 through M. The merged firm picks output $Q_M = q_1 + q_2 + \ldots + q_M$. The profit maximization problem for the merged firm is given by

$$
\begin{aligned}
\pi_M &= \max_{Q_M} \left[p\, Q_M - c Q_M \right] \\
&= \max_{q_M} \left[(A - B\, Q_{-M} - B\, Q_M)\, Q_M - c\, Q_M \right] \\
&= \max_{Q_M} \left[A\, Q_M - B\, Q_{-M}\, Q_M - B\, Q_M^2 - c\, Q_M \right]
\end{aligned} \tag{8}
$$

The profit maximization problem for each of the remaining independent firms is given by

$$
\begin{aligned}
\pi_i &= \max_{q_i} \left[p\, q_i - c q_i \right] \\
&= \max_{q_i} \left[(A - B\, Q_{-i} - B\, q_i)\, q_i - c\, q_i \right] \\
&= \max_{q_i} \left[A\, q_i - B\, Q_{-i}\, q_i - B\, q_i^2 - c\, q_i \right]
\end{aligned} \tag{9}
$$

Here Q_{-i} denotes the aggregate output of all N-M firms other than the i^{th} firm including the output of the merged firm Q_M. After the merger the *merged firm is just like any one of the other m+1, m+2, ... , N in the industry*.

2.1.5. *Industry Equilibrium.* Because the problem is analogous to the one in equation 2, we can use the results and intuition from that problem to obtain the equilibrium quantities

$$
\begin{aligned}
Q_M^* &= \frac{(A - c)}{(N - M + 2)\, B} \\
q_i^* &= \frac{(A - c)}{(N - M + 2)\, B}, \; i \notin M \\
Q^* &= Q_M^* + (N - M)\,(q_i^*) = (N - M + 1) \left(\frac{(A - c)}{(N - M + 2)\, B} \right) \\
&= \left(\frac{(N - M + 1)\,(A - c)}{(N - M + 2)\, B} \right)
\end{aligned} \tag{10}
$$

The equilibrium price is given by

$$
\begin{aligned}
p^* &= A - BQ^* \\
&= A - B \left(\frac{(N - M + 1)\,(A - c)}{(N - M + 2)\, B} \right) \\
&= \frac{(N - M + 2)\, A}{N - M + 2} - \frac{(N - M + 1)\, A}{(N - M + 2)} - \frac{(N - M + 1)\, c}{N - M + 2} \\
&= \frac{A}{N - M + 2} + \frac{N - M + 1}{N - M + 2}\, c
\end{aligned} \tag{11}
$$

Similarly profits are given by

$$
\begin{aligned}
\pi_M = \pi_i &= p^* q_M^* - c q_M^* \\
&= \left(\frac{A}{N - M + 2} + \frac{N - M + 1}{N - M + 2}\, c \right) Q_M^* - c\, Q_M^* \\
&= \left[\frac{A}{N - M + 2} + \left(\frac{N - M + 1}{N - M + 2} - 1 \right) c \right] \left[\frac{A - c}{(N - M + 2)\, B} \right] \\
&= \left[\frac{A - c}{N - M + 2} \right] \left[\frac{A - c}{(N - M + 2)\, B} \right] \\
&= \frac{(A - c)^2}{(N - M + 2)^2\, B}
\end{aligned} \tag{12}
$$

Equations 7 and 12 allow us to compare the profit of the non-merging firms before and after the others merge. It is clear that profits are larger after the merger for the firms who do not merge because the denominator in equation 12 is smaller than the denominator in equation 7 if M is larger than 1. Now consider the firms who merged. Prior to the merger they each earned the profit in equation 7. Their total profit is given by M times this amount or

$$
\pi_M(pre - merger) \equiv M \; \frac{(A - c)^2}{(N + 1)^2\, B} \tag{13}
$$

After the merger they get the amount in equation 12. For profits to be higher it must be that

$$\frac{(A - c)^2}{(N - M + 2)^2 B} \geq M \left(\frac{(A - c)^2}{B(N + 1)^2} \right)$$
$$\Rightarrow (N + 1)^2 \geq M(N - M + 2)^2 \tag{14}$$

This condition is very hard to satisfy unless M is very close to N. For example if N = 3 and M = 2, equation 13 would imply 16 $\geq$ 18 which is false. Similarly if N = 10 and M = 7, equation 14 would imply 121 $\geq$ 175. If N = 10 and M = 9, equation 14 implies 121 $\geq$ 81. The tendency seems to be for mergers to pay off only when the merged firm is a large portion of the industry. For example if half the firms merge so that 2M = N equation 14 implies

$$(2M + 1)^2 \geq (M)(2M\ M + 2)^2$$
$$\Rightarrow 4M^2 + 4M + 1 \geq M^3 + 4M^2 + 4M \tag{15}$$
$$\Rightarrow 1 \geq M$$

It turns out that M must comprise greater than 80 per cent of the market for the merger to lead to higher profits.

2.1.6. *Criticism of this model.* The key result of the model is that the merged firm ends up being just another player in a Cournot game and gains no advantage due to the merger. This is clearest in a triopoly setting where two firms merge. The merged firms would then share half of the market.

2.2. **A Stackelberg Model of Mergers.**

2.2.1. *Assumptions.*

1. N $\geq$ 2 firms are in the industry. Two firms merge so that M = 2. Let these be denoted firm 1 and firm 2. Denote the output of these two firms by Q_M and the output of the other N-M firms as Q_I. I is then equal to N-2 for this case.
2. Identical and constant costs for each firm given by $C(q_j) = cq_j$, j = 1,2, ... , N
3. Linear inverse demand given by

$$\begin{aligned} p &= A - BQ \\ &= A - B(q_1 + \cdots + q_{i-1} + q_i + q_{i+1} + \cdots + q_N) \\ &= A - BQ_I - BQ_M \end{aligned} \tag{16}$$

   where Q_I is the output of all the independent firms. For one of the independent firms residual inverse demand is given by

$$\begin{aligned} p &= A - BQ_{I-i} - Bq_i - BQ_M\,, \quad Q_{I-i} = \Sigma_{\substack{j \notin M \\ j \neq i}} q_j \\ &= A - BQ_{-i} - Bq_i, \quad Q_{-i} = \Sigma_{j \neq i}\, q_j \end{aligned} \tag{17}$$

4. Industry supply given by $Q = Q_I + Q_M$, $Q_I = \Sigma_{j \notin M}\, q_j$
5. The merged firm acts as a Stackelberg leader. The other firms act as followers.

2.2.2. *The profit maximization problem for a non-merged follower firm.* The follower firms will maximize profits taking the output of the merged firms as given. Because the merged firm moves first, these firms can take this quantity as given. This makes the problem a sequential as opposed to a simultaneous move game. Profit for the j^{th} firm is given by

$$\begin{aligned}\pi_j &= \max_{q_j} \left[p\, q_j - c q_j \right] \\ &= \max_{q_j} \left[(A - B\, q_j - B\, Q_{I-j} - B Q_M)\, q_j - c\, q_j \right] \\ &= \max_{q_j} \left[A\, q_j - B\, q_j\, Q_{I-j} - B Q_M q_j - B\, q_j^2 - c\, q_j \right]\end{aligned} \tag{18}$$

Maximizing profit we obtain

$$\begin{aligned}\frac{d\pi_j}{dq_j} &= A - B\, Q_{I-j} - B\, Q_M - 2\, B\, q_j - c = 0 \\ \Rightarrow\ 2\, B\, q_j &= A - B\, Q_{I-j} - B\, Q_M - c \\ \Rightarrow\ q_j^* &= \frac{A - c}{2\, B} - \frac{Q_M}{2} - \frac{Q_{I-j}}{2}\end{aligned} \tag{19}$$

This is the reaction function for firm j. The non-merged firm acts like any other Cournot firm (Stackelberg follower) but now takes the output of both merged and other non-merged firms into account. Because all these firms are the same we get their total supply by multiplication

$$\begin{aligned}Q_{I-j} &= (I - 1)\, q_j \\ \Rightarrow\ q_j &= \frac{Q_{I-j}}{(I - 1)}\end{aligned} \tag{20}$$

If we solve equations 19 and 20 together we obtain

$$\begin{aligned}q_j^* &= \frac{A - c}{2\, B} - \frac{Q_M}{2} - \frac{(I - 1)\, q_j^*}{2} \\ \Rightarrow\ \frac{2\, q_j^*}{2} + \frac{(I - 1)\, q_j^*}{2} &= \frac{A - c}{2\, B} - \frac{Q_M}{2} \\ \Rightarrow\ (I + 1)\, q_j^* &= \frac{A - c}{B} - Q_M \\ \Rightarrow\ q_j^* &= \frac{A - c}{B\,(I + 1)} - \frac{Q_M}{(I + 1)}\end{aligned} \tag{21}$$

Aggregate output by these firms is given by

$$Q_I = \frac{I}{(I + 1)} \left(\frac{A - c}{B} - Q_M \right) \tag{22}$$

2.2.3. *The profit maximization problem for the merged firm.* The merged firm realizes that once it chooses its output, Q_M, the other firms will use equation 21 to pick their optimal output $q_j = R_j(Q_M)$. Consider the residual inverse demand for the merged firms which is given by

$$p = A - B\, Q_I - B\, Q_M \tag{23}$$

Now substitute in the optimal response function for the independent firms to obtain

$$p = A - B\left(\frac{I}{(I+1)}\right)\left(\frac{A-c}{B} - Q_M\right) - B\,Q_M \tag{24}$$

Profit for the merged firms is given by

$$\begin{aligned}
\pi_M &= \max_{Q_M}\left[p\,Q_M - cQ_M\right] \\
&= \max_{Q_M}\left[\left(A - B\left(\frac{I}{(I+1)}\right)\left(\frac{A-c}{B} - Q_M\right) - B\,Q_M\right)Q_M - c\,Q_M\right] \\
&= \max_{Q_M}\left[\left(\frac{(I+1)\,A}{I+1} - \left(\frac{I\,(A-c)}{(I+1)}\right) + \frac{B\;I\,Q_M}{I+1} - \frac{B\,(I+1)\,Q_M}{I+1}\right)Q_M - \frac{c\,(I+1)Q_M}{I+1}\right] \\
&= \max_{Q_M}\left[\left(\frac{A - c - B\,Q_M}{I+1}\right)Q_M\right] \\
&= \max_{Q_M}\left[\left(\frac{A\,Q_M - c\,Q_M - B\,Q_M^2}{I+1}\right)\right]
\end{aligned} \tag{25}$$

If we now maximize equation 25 with respect to Q_M, we obtain

$$\begin{aligned}
\frac{d\pi_M}{dq_M} &= \frac{d\left[\left(\frac{A\,Q_M - c\,Q_M - B\,Q_M^2}{I+1}\right)\right]}{d\,Q_M} = 0 \\
\Rightarrow A - c - 2BQ_M &= 0 \\
\Rightarrow Q_M^* &= \frac{A-c}{2B}
\end{aligned} \tag{26}$$

2.2.4. *Equilibrium output for an independent firm.* The equilibrium output of an independent firm is then given by substituting 26 into 21 as follows

$$\begin{aligned}
q_j^* &= \frac{A-c}{B\,(I+1)} - \frac{Q_M}{(I+1)} \\
&= \frac{A-c}{B\,(I+1)} - \frac{A-c}{2B\,(I+1)} \\
&= \frac{A-c}{2B\,(I+1)} \\
&= \frac{A-c}{2B\,(N-1)}, \quad if\ M = 2
\end{aligned} \tag{27}$$

Total output of the independent firms is given by

$$\begin{aligned}
Q_i^* &= \frac{I\,(A-c)}{(I+1)\;2B} \\
&= \frac{(N-2)\,(A-c)}{(N-1)\;2B}, \quad if\ M = 2
\end{aligned} \tag{28}$$

2.2.5. *Equilibrium for the industry.* Total industry output is given by adding equations 26 and 28 together to obtain

$$\begin{aligned} Q^* &= Q_i^* + Q_M^* \\ &= \frac{I\,(A - c)}{(I + 1)\;2B} + \frac{A - c}{2B} \\ &= \frac{(2I + 1)\,(A - c)}{(I + 1)\;2B} \end{aligned} \tag{29}$$

The industry equilibrium price is given by

$$\begin{aligned} p &= A - B\,Q^* \\ &= A - B\left[\frac{(2I + 1)\,(A - c)}{(I + 1)\;2B}\right] \\ &= \frac{2\,(I + 1)\,A}{2\,(I + 1)} - \left[\frac{(2I + 1)\,(A - c)}{(I + 1)\;2}\right] \\ &= A - B\left[\frac{(2I + 1)\,(A - c)}{(I + 1)\;2B}\right] \\ &= \frac{A + c\,(2I + 1)}{2\,(I + 1)} \end{aligned} \tag{30}$$

The merged firm has profit

$$\begin{aligned} \pi_M &= pQ_M - c\,Q_M \\ &= \left(\frac{A + c\,(2I + 1)}{2\,(I + 1)}\right)\left[\frac{A - c}{2B}\right] - c\left[\frac{A - c}{2B}\right] \\ &= \left(\frac{A + 2I\,c + c - 2c(I + 1)}{2\,(I + 1)}\right)\left[\frac{A - c}{2B}\right] \\ &= \left(\frac{A - c}{2\,(I + 1)}\right)\left[\frac{A - c}{2B}\right] \\ &= \left(\frac{(A - c)^2}{4\,B(I + 1)}\right) \end{aligned} \tag{31}$$

Each of the independent firms has profit equal to

$$
\begin{aligned}
\pi_j &= pq_j - c\,q_j \\
&= \left(\frac{A + c\,(2I + 1)}{2\,(I + 1)} \right) \left(\frac{A - c}{2B\,(I + 1)} \right) - c \left[\frac{A - c}{2B\,(I + 1)} \right] \\
&= \left(\frac{A - c}{2B\,(I + 1)} \right) \left(\frac{A + c\,(2I + 1)}{2\,(I + 1)} - \frac{2\,c\,(I + 1)}{2\,(I + 1)} \right) \\
&= \left(\frac{A - c}{2B\,(I + 1)} \right) \left(\frac{A + 2cI + c - 2cI - 2c)}{2\,(I + 1)} \right) \\
&= \left(\frac{A - c}{2\,(I + 1)} \right) \left[\frac{A - c}{2B\,(I + 1)} \right] \\
&= \left(\frac{(A - c)^2}{4\,B(I + 1)^2} \right)
\end{aligned}
\tag{32}
$$

The merged firm has profits that are much larger than any individual independent firm. We can also compare the profit of this merged firm to the combined profits of the M firms when they were members of the Cournot equilibrium before the merger. This profit is given by equation 7. We repeat it here for convenience

$$
\begin{aligned}
\pi_k &= p^* q_k^* - c\,q_k^* \\
&= \frac{(A - c)^2}{(N + 1)^2\,B}
\end{aligned}
\tag{7'}
$$

For the combined profit after merger to be larger than combined profit before merger it must be true that

$$
\begin{aligned}
\left(\frac{(A - c)^2}{4\,B(I + 1)} \right) &\geq \frac{M\,(A - c)^2}{(N + 1)^2\,B} \\
\Rightarrow \left(\frac{1}{4\,(N - M + 1)} \right) &\geq \frac{M}{(N + 1)^2} \\
\Rightarrow (N + 1)^2 &\geq 4M\ ((N - M + 1)
\end{aligned}
\tag{33}
$$

This equation is satisfied exactly if N = 3 and M = 2. It is true for any N greater than 3 if M = 2.

Now consider the profits of an independent firm. They earn the same amount as the non-merged firm before the merger. For them to earn less after the merger it must be true that

$$
\begin{aligned}
\left(\frac{(A - c)^2}{4\,B(I + 1)^2} \right) &\leq \frac{(A - c)^2}{(N + 1)^2\,B} \\
\Rightarrow \left(\frac{1}{4\,(N - M + 1)^2} \right) &\leq \frac{1}{(N + 1)^2} \\
\Rightarrow (N + 1)^2 &\leq 4\,(N - M + 1)^2
\end{aligned}
\tag{34}
$$

As long as N $\geq$ 4 this will hold true, and non-merging firms will be worse off unless M is also very large.

2.2.6. *Prices and welfare.* It is useful to compare the price cost margin in the Stackelberg merger case with that prevailing in the initial Cournot equilibrium. The margin in the initial case is computed using equation 6. The margin in the Stackelberg case is computed using equation 30. The margins are as follows

$$\begin{aligned} pcm(Cournot) &= \frac{A}{N+1} + \frac{N}{N+1}c - \frac{(N+1)c}{(N+1)} \\ pcm(Stackelberg) &= \frac{A + c(2I+1)}{2(I+1)} - \frac{2(I+1)c}{2(I+1)} \end{aligned} \tag{35}$$

We can rewrite I in terms of M and N (I = M - N) and then note that prices will be lower with the Stackelberg merger if

$$\begin{aligned} pcm(Stackelberg) = \frac{A + c(2(N-M)+1)}{2((N-M)+1)} - \frac{2((N-M)+1)c}{2(N-M+1)} &\le \frac{A}{N+1} + \frac{N}{N+1}c - \frac{(N+1)c}{(N+1)} = pcm(Cournot) \\ \frac{A + 2cN - 2cM + c - 2Nc + 2Mc - 2c}{2((N-M)+1)} &\le \frac{A + Nc - Nc - c}{N+1} \\ \frac{A-c}{2((N-M)+1)} &\le \frac{A-c}{N+1} \\ N+1 &\le 2(N-M+1) \\ \Rightarrow 2M - 1 &\le N \end{aligned} \tag{36}$$

With M = 2, this implies that for N $\geq$ 3, the price cost margin is lower with the Stackelberg merger. Thus consumers seem to be better off. If this is really the case, there would seem to be no reason for anti-trust authorities to ever oppose mergers.

2.3. **Multiple mergers - several leaders and followers.**

2.3.1. *Assumptions.*

1. N firms in the industry
2. L leader firms - each formed from a merger of two previously independent firms. These are numbered 1, 2, ..., L
3. F = N-L follower firms. These are numbered L+1, L+2, ..., N
4. Followers take total output of leader firms as given and then individually maximize profit.
5. Leader firms engage in Cournot competition vis-a vis each other taking into account the actions of the follower firms.

6. Notation for quantities (number of firms in brackets)

$$
\begin{aligned}
Q \; &= total\ market\ quantity\ supplied\ and\ demanded\ [N] \\
q_f \; &= output\ of\ one\ independent\ firm\ (follower)\ [1] \\
q_\ell \; &= output\ of\ one\ of\ the\ merged\ firms\ (leader)\ [1] \\
Q_L \; &= \Sigma_{\ell=1}^{L}\, q_\ell \;=\; aggregate\ output\ of\ leader\ firms\ [L] \\
Q_F \; &= \Sigma_{f=1}^{F}\, q_f \;=\; aggregate\ output\ of\ follower\ firms\ [F] \\
Q_{L-\ell} \; &= \Sigma_{\substack{k \neq \ell \\ k \epsilon [1,L]}}\, q_k = \; aggregate\ output\ of\ leader\ firms\ except\ \ell\ [L-1 = N-F-1] \\
Q_{F-f} \; &= \Sigma_{\substack{j \neq f \\ j \epsilon [L+1,N]}}\, q_j \;=\; aggregate\ output\ of\ follower\ firms\ except\ f\ [F-1 = N-L-1]
\end{aligned}
\tag{37}
$$

7. Cost of production of quantity q_k = cq_k.
8. Inverse demand is given by

$$
\begin{aligned}
p \; &= A \,-\, BQ \\
&= A \,-\, BQ_L \,-\, BQ_F \\
&= A \,-\, BQ_L \,-\, BQ_{F-f} \,-\, Bq_f \\
&= A \,-\, BQ_{L-\ell} \,-\, Bq_\ell \,-\, BQ_F \\
&= A \,-\, BQ_{L-\ell} \,-\, Bq_\ell \,-\, BQ_{F-f} \,-\, Bq_f
\end{aligned}
\tag{38}
$$

2.4. **Profit maximization for a representative follower firm.** The profit maximization problem for a typical follower firm (taking Q_L and Q_{F-f} as given) is

$$
\begin{aligned}
\pi_f^F \; &= \; \max_{q_f} \, [p \, q_f \,-\, C(q_f)] \\
&= \; \max_{q_f} \, [(A \,-\, B\,(q_f \,+\, Q_{F-f} \,+\, Q_L)\, q_f \,-\, c\, q_f] \\
&= \; \max_{q_f} \, [A\, q_f - \, Bq_f^2 \,-\, BQ_{F-f}\, q_f \,-\, BQ_L\, q_f \,-\, c\, q_f]
\end{aligned}
\tag{39}
$$

Carrying out the maximization will give

$$
\begin{aligned}
\frac{d\pi_f^F}{dq_f} \; &= \frac{d\,[A\, q_f \,-\, Bq_f^2 \,-\, BQ_{F-f}\; q_f \,-\, B\, Q_L\, q_f \,-\, c\, q_f]}{dq_f} \;=\; 0 \\
\Rightarrow \; A - 2B\, q_f &- \, BQ_{F-f} \,-\, B\, Q_L \,-\, c \;=\; 0 \\
\Rightarrow \; q_f \; &= \frac{A \,-\, c}{2B} \,-\, \frac{Q_L}{2} \,-\, \frac{Q_{F-f}}{2}
\end{aligned}
\tag{40}
$$

2.4.1. *Equilibrium response for the follower firms.* The follower firms are all symmetric and so each one will have the same response function as in equation 40. There are (N-L-1) firms in the group Q_{F-f}. Total supply by these firms is then given by

$$
Q_{F-f} \;=\; [N \,-\, L \,-\, 1]\, q_f \tag{41}
$$

If we substitute equation 41 into equation 40 we obtain

$$
\begin{aligned}
q_f^* &= \frac{A - c}{2B} - \frac{Q_L}{2} - \frac{Q_{F-f}}{2} \\
&= \frac{A - c}{2B} - \frac{Q_L}{2} - \frac{(N - L - 1)\, q_f}{2} \\
\Rightarrow \frac{2q_f + Nq_f - L\, q_f - q_f}{2} &= \frac{A - c}{2B} - \frac{Q_L}{2} \\
\Rightarrow \frac{(N - L + 1)}{2}\, q_f &= \frac{A - c}{2B} - \frac{Q_L}{2} \\
\Rightarrow q_f^* &= \frac{(A - c)}{B\,(N - L + 1)} - \frac{Q_L}{(N - L + 1)}
\end{aligned} \tag{42}
$$

Total supply for the F follower firms is then given by [N-L]q_f^*. This gives

$$
Q_F = \frac{(N - L)\,(A - c)}{B\,(N - L + 1)} - \frac{(N - L)\, Q_L}{(N - L + 1)} \tag{43}
$$

2.4.2. *Profit maximization for a representative leader firm.* The profit maximization problem for a typical leader firm (taking $Q_{L-\ell}$ and Q_F as given) is

$$
\begin{aligned}
\pi_\ell^L &= \max_{q_\ell} \left[p\, q_\ell - C(q_\ell)\right] \\
&= \max_{q_\ell} \left[(A - B\,(q_\ell + Q_F + Q_{L-\ell})\, q_\ell - c\, q_\ell\right] \\
&= \max_{q_\ell} \left[A\, q_\ell - Bq_\ell^2 - BQ_F\, q_\ell - B\, Q_{L-\ell}\, q_\ell - c\, q_\ell\right]
\end{aligned} \tag{44}
$$

If we now substitute from 43 into 44 we obtain

$$
\begin{aligned}
\pi_\ell^L &= \max_{q_\ell} \left[A\, q_\ell - B q_\ell^2 - B Q_F\, q_\ell - B\, Q_{L-\ell}\, q_\ell - c\, q_\ell\right] \\
&= \max_{q_\ell} \left[A\, q_\ell - B q_\ell^2 - B \left[\frac{(N-L)\,(A-c)}{B\,(N-L+1)} - \frac{(N-L)\,Q_L}{(N-L+1)}\right] q_\ell - B\, Q_{L-\ell}\, q_\ell - c\, q_\ell\right] \\
&= \max_{q_\ell} \left(q_\ell \left[A - c - B q_\ell - \frac{B\,(N-L)}{B\,(N-L+1)}\,(A-c) + B\,\frac{(N-L)\,Q_L}{(N-L+1)} - B\,Q_{L-\ell}\right]\right) \\
&= \max_{q_\ell} \left(q_\ell \left[A - c - B Q_L - \frac{B\,(N-L)}{B\,(N-L+1)}\,(A-c) + B\,\frac{(N-L)\,Q_L}{(N-L+1)}\right]\right) \\
&= \max_{q_\ell} \left(q_\ell \left[\frac{(N-L+1)\,(A-c)}{(N-L+1)} - \frac{B\,(N-L+1)\,Q_L}{(N-L+1)} - \frac{(N-L)}{(N-L+1)}\,(A-c) + B\,\frac{(N-L)\,Q_L}{(N-L+1)}\right]\right) \\
&= \max_{q_\ell} \left(q_\ell \left[\frac{(N-L+1)\,(A-c) - B\,(N-L+1)\,Q_L - (N-L)\,(A-c) + B\,(N-L)\,Q_L}{(N-L+1)}\right]\right) \\
&= \max_{q_\ell} \left(q_\ell \left[\frac{(A-c) - B\,Q_L}{(N-L+1)}\right]\right) \\
&= \max_{q_\ell} \left(q_\ell \left[\frac{(A - c - B\,Q_{L-\ell} - B\,q_\ell}{(N-L+1)}\right]\right) \\
&= \max_{q_\ell} \left(\frac{(A q_\ell - c q_\ell - B\,Q_{L-\ell} q_\ell - B\,q_\ell^2}{(N-L+1)}\right)
\end{aligned}
\tag{45}
$$

Carrying out the maximization will give

$$
\begin{aligned}
\frac{d\pi_\ell^L}{q_\ell} &= \frac{d\left(\frac{(A q_\ell - c q_\ell - B\,Q_{L-\ell} q_\ell - B\,q_\ell^2}{(N-L+1)}\right)}{d q_\ell} = 0 \\
&\Rightarrow A - c - B Q_{L-\ell} - 2\,B\,q_\ell = 0 \\
&\Rightarrow q_\ell = \frac{A-c}{2B} - \frac{Q_{L-\ell}}{2}
\end{aligned}
\tag{46}
$$

2.4.3. *Equilibrium response for the leader firms.* The leader firms are all symmetric and so each one will have the same response function as in equation 46. There are (L-1) firms in the group $Q_{l-\ell}$. Total supply by these firms in then given by

$$
Q_{L-\ell} = [L - 1]\, q_\ell \tag{47}
$$

If we substitute equation 47 into equation 46 we obtain

$$
\begin{aligned}
q_\ell^* &= \frac{A - c}{2B} - \frac{Q_{L-\ell}}{2} \\
&= \frac{A - c}{2B} - \frac{(L - 1)\, q_\ell}{2} \\
\Rightarrow \frac{2q_\ell + (L - 1)\, q_\ell}{2} &= \frac{A - c}{2B} \\
\Rightarrow \frac{(L + 1)\, q_\ell}{2} &= \frac{A - c}{2B} \\
\Rightarrow q_\ell^* &= \frac{(A - c)}{B\,(L + 1)}
\end{aligned}
\tag{48}
$$

Total supply for the L leader firms is then given Lq_ℓ^*. This gives

$$
Q_L = \frac{L\,(A - c)}{B\,(L + 1)}
\tag{49}
$$

We can now find q_f and Q_F by substituting equation 49 into equations 42 and 43.

$$
\begin{aligned}
q_f^* &= \frac{(A - c)}{B\,(N - L + 1)} - \frac{Q_L}{(N - L + 1)} \\
&= \frac{(A - c)}{B\,(N - L + 1)} - \frac{L\,(A - c)}{B\,(L + 1)\,(N - L + 1)} \\
&= \frac{(L + 1)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} - \frac{L\,(A - c)}{B\,(L + 1)\,(N - L + 1)} \\
&= \frac{(A - c)}{B\,(L + 1)\,(N - L + 1)}
\end{aligned}
\tag{50}
$$

Total supply by the follower firms is

$$
Q_F^* = \frac{(N - L)\,(A - c)}{B\,(L + 1)\,(N - L + 1)}
\tag{51}
$$

2.4.4. *Industry output.* Total industry output is given by adding Q_L and Q_F to obtain

$$
\begin{aligned}
Q^* &= Q_F^* + Q_L^* \\
&= \frac{(N - L)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} + \frac{L\,(A - c)}{B\,(L + 1)} \\
&= \frac{(N - L)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} + \frac{L\,(N - L + 1)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} \\
&= \frac{(N - L + LN - L^2 + L)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} \\
&= \frac{(N + LN - L^2)\,(A - c)}{B\,(L + 1)\,(N - L + 1)}
\end{aligned}
\tag{52}
$$

We now have the equilibrium quantities of each leader firm, each follower firm, the aggregate for the leaders and the followers, and the total quantity.

2.4.5. *Price and price cost margin.* The market equilibrium price is given by

$$\begin{aligned}
p^* &= A - BQ^* \\
&= A - B\left[\frac{(N + LN - L^2)(A - c)}{B(L + 1)(N - L + 1)}\right] \\
&= \frac{(L + 1)(N - L + 1)A}{(L + 1)(N - L + 1)} - \frac{(N + LN - L^2)(A - c)}{(L + 1)(N - L + 1)} \\
&= \frac{A(LN - L^2 + L + N - L + 1 - N - LN + L^2) + c(N + LN - L^2)}{(L + 1)(N - L + 1)} \\
&= \frac{A + c(N + LN - L^2)}{(L + 1)(N - L + 1)}
\end{aligned} \tag{53}$$

The price cost margin is given by

$$\begin{aligned}
p^* - c &= \frac{A + c(N + LN - L^2)}{(L + 1)(N - L + 1)} - \frac{(L + 1)(N - L + 1)c}{(L + 1)(N - L + 1)} \\
&= \frac{A + cN + cLN - cL^2 - cLN + cL^2 - cN - c)}{(L + 1)(N - L + 1)} \\
&= \frac{A - c}{(L + 1)(N - L + 1)}
\end{aligned} \tag{54}$$

2.5. **Analysis of equilibrium in horizontal merger models.**

2.5.1. *Individual firms in the multiple firm merger model with Stackelberg behavior.* It is clear by comparing equations 48 and 50 that the leader firms produce more than the follower firms.

2.5.2. *Comparison of Stackelberg merger equilibrium with several leaders with regular Cournot equilibrium.* The Cournot equilibrium from equation 6 and the Stackelberg one from equation 52 are

$$\begin{aligned}
Q_C^* &= \frac{N(A - c)}{(N + 1)B} \\
Q_{SM}^* &= \frac{(N + LN - L^2)(A - c)}{B(L + 1)(N - L + 1)}
\end{aligned} \tag{55}$$

If L = 0, they are the same. If L = 1, we have

$$\begin{aligned}
Q_C^* &= \frac{N(A - c)}{(N + 1)B} \\
Q_{SM}^* &= \frac{(N + N - 1)(A - c)}{B(1 + 1)(N - 1 + 1)} \\
&= \frac{(2N - 1)(A - c)}{2BN}
\end{aligned} \tag{56}$$

To show that $Q_{SM} \geq Q_c$ we must show that

$$
\begin{aligned}
(2N - 1)(N + 1) &\geq 2N^2 \\
\Rightarrow (2N^2 + N - 1) &\geq 2N^2 \\
\Rightarrow N - 1 &\geq 0
\end{aligned}
$$

This is always true if N $\geq$ 1. So the expression is true for L = 1. In general the merger solution will have a larger output if

$$
\begin{aligned}
\frac{(N + NL - L^2)(A - c)}{B(L + 1)(N - L + 1)} &\geq \frac{N(A - c)}{(N + 1)B} \\
\Rightarrow \frac{(N + NL - L^2)}{(L + 1)(N - L + 1)} &\geq \frac{N}{(N + 1)} \\
\Rightarrow N^2 + N^2L - NL^2 + N + NL - L^2 &\geq N^2L - NL^2 + NL + N^2 - NL + N \\
\Rightarrow -L^2 &\geq -NL \\
\Rightarrow L^2 &\leq NL \\
\Rightarrow L &\leq N
\end{aligned}
\tag{57}
$$

This will always be the case. Thus the quantity will always be higher with the mergers.

2.5.3. *Comparison of profits for leader and follower firms.* Profits for a follower firm are obtained by multiplying the price-cost margin by output. This will yield

$$
\begin{aligned}
\pi_f^F &= q_f\ (p^* - c) \\
&= \left(\frac{(A - c)}{B(L + 1)(N - L + 1)} \right) \left(\frac{A - c}{(L + 1)(N - L + 1)} \right) \\
&= \left(\frac{(A - c)^2}{B(L + 1)^2(N - L + 1)^2} \right)
\end{aligned}
\tag{58}
$$

Profits for a leader firm are obtained by multiplying the price-cost margin by output. This will yield

$$
\begin{aligned}
\pi_\ell^L &= q_\ell (p^* - c) \\
&= \left(\frac{(A - c)}{B(L + 1)} \right) \left(\frac{A - c}{(L + 1)(N - L + 1)} \right) \\
&= \left(\frac{(A - c)^2}{B(L + 1)^2(N - L + 1)} \right)
\end{aligned}
\tag{59}
$$

It is very clear comparing equations 58 and 59 that leaders are more profitable than followers. Thus two firms who are followers may want to consider merging. But if they merge the number of leaders will rise, the number of followers will fall and the industry price cost margin will change. What we need to discover is the profit of the currently independent firms in the new equilibrium after merger.

2.5.4. *Analysis of incentives to merge.* In the new equilibrium N will be one less, and L will be one more. Now consider equation 59 with N-1 replacing N and L+1 replacing L. This will give

$$
\begin{aligned}
\pi_\ell^L(N - 1, L + 1) &= \left(\frac{(A - c)^2}{B(L + 2)^2((N - 1) - (L + 1) + 1)} \right) \\
&= \left(\frac{(A - c)^2}{B(L + 2)^2(N - L - 1)} \right)
\end{aligned}
\tag{60}
$$

We then compare this to 2 times equation 58. If it is larger then there is an incentive to merge. Writing out the inequality gives

$$\left(\frac{(A - c)^2}{B\,(L + 2)^2\,(N - L - 1)} \right) \geq 2 \left(\frac{(A - c)^2}{B\,(L + 1)^2\,(N - L + 1)^2} \right) \tag{61}$$

We can then simplify to obtain

$$\begin{aligned}
2\,(L + 2)^2\,(N - L - 1) &\leq (L + 1)^2\,(N - L + 1)^2 \\
\Rightarrow (2L^2 + 8L + 8)\,(N - L - 1) &\leq (L^2 + 2L + 1)\,(N^2 - 2NL + L^2 + 2(N - L) + 1) \\
\Rightarrow 2L^2N - 2L^3 + 8LN - 10L^2 + 8N - 16L - 8 &\leq (L^2 + 2L + 1)\,((N^2 - 2NL + 2N + L^2 - 2L + 1) \\
\Rightarrow 2L^2N - 2L^3 - 10L^2 + 8LN - 16L + 8N - 8 &\leq L^2\,N^2 - 2NL^3 + 2NL^2 + L^4 - 2L^3 + L^2 \\
&\quad + 2L\,N^2 - 4NL^2 + 4NL + 2\,L^3 - 4L^2 + 2L \\
&\quad + N^2 - 2NL + 2N + L^2 - 2L + 1 \\
\Rightarrow 2L^2N - 2L^3 - 10L^2 + 8LN - 16L + 8N - 8 &\leq L^2\,N^2 - 2NL^3 + L^4 - 2NL^2 - 2L^2 + 2L\,N^2 + 2NL + N^2 + 2N + 1
\end{aligned} \tag{62}$$

This expression does not seem to simplify but numerical simulation shows that it is always true so that there is always an incentive to merge. Thus once mergers begin, they will tend to continue.

2.5.5. *Social costs of mergers.* Another question is whether the mergers are in the best interest of society. We can see from equation 54 that the price-cost margin depends on the number of firms and the number of firms in the leadership position (L). The price cost margin will fall as the industry output rises. Consider now the total industry output in the Stackelberg merger case when two more firms merge. We can consider total output in equation 52 with the output when N decreases by one and L increases by one. We proceed as follows by rewriting equation 52 and then the inequality

$$\begin{aligned}
Q^*(N-1,\, L+1) = \frac{((N-1) + (L+1)\,(N-1) - (L + 1)^2)\,(A - c)}{B\,(L + 2)\,((N-1) - (L+1) + 1)} &\geq \frac{(N + LN - L^2)\,(A - c)}{B\,(L + 1)\,(N - L + 1)} = Q^*(N,\, L) \\
\Rightarrow \frac{((N-1) + (L+1)\,(N-1) - (L + 1)^2)}{(L + 2)\,((N - L - 1)} &\geq \frac{N + LN - L^2}{(L + 1)\,(N - L + 1)} \\
\Rightarrow \frac{(L + 1)\,(N - L + 1)\,((N-1) + (L+1)\,(N-1) - (L + 1)^2)}{(L + 1)\,(N - L + 1)\,(L + 2)\,((N - L - 1)} &\geq \frac{(L + 2)\,((N - L - 1)\,(N + LN - L^2\,)}{(L + 2)\,((N - L - 1)\,(L + 1)\,(N - L + 1)}
\end{aligned} \tag{63}$$

The denominator of equation 63 will always be positive because N > L. Thus we need to show that the numerator on the left is larger than the numerator on the right. We will first simplify the left hand side and then the right hand side. Then we will put the information together. First for the left hand side

$$\begin{aligned}
(L + 1)\,(N - L + 1)\,\big((N-1) + (L+1)\,(N-1) - (L + 1)^2\big) &= (LN - L^2 + N + 1)\,\big(2N + LN - L^2 - 3L - 3\big) \\
&= (\,2LN^2 + L^2N^2 - L^3N - 3L^2N - 3LN\,) \\
&\quad + (\,-2L^2N - L^3N + L^4 + 3L^3 + 3L^2\,) \\
&\quad + (\,2N^2 + LN^2 - L^2N - 3LN - 3N\,) \\
&\quad + (\,2N + LN - L^2 - 3L - 3\,) \\
&= L^4 + 3L^3 + 2L^2 - 3L - 2L^3N - 6L^2N + L^2N^2 \\
&\quad - 5LN + 3LN^2 + 2N^2 - N - 3
\end{aligned} \tag{64}$$

Now for the right hand side

$$
\begin{aligned}
(L+2)(N-L-1)(N+LN-L^2) &= (N+LN-L^2)(LN-L^2-3L+2N-2) \\
&= (LN^2-L^2N-3LN+2N^2-2N) \\
&+ (L^2N^2-L^3N-3L^2N+2LN^2-2LN) \\
&+ (-L^3N+L^4+3L^3-2L^2N+2L^2) \\
&= L^4+3L^3+2L^2-2L^3N-6L^2N \\
&+ L^2N^2+3LN^2-5LN+2N^2-2N
\end{aligned} \tag{65}
$$

Now take the right hand side of equations 64 and 65 and substitute them in to rewrite the numerators of equation 63 as follows

$$
\begin{aligned}
&L^4+3L^3+2L^2-3L-2L^3N-6L^2N+L^2N^2-5LN+3LN^2+2N^2-N-3 \geq \\
&L^4+3L^3+2L^2-2L^3N-6L^2N+L^2N^2-5LN+3LN^2+2N^2-2N
\end{aligned} \tag{66}
$$

Canceling like terms we obtain

$$
\begin{aligned}
-3L-N-3 &\geq -2N \\
\Rightarrow N &\geq 3L+3
\end{aligned} \tag{67}
$$

or alternatively

$$
\begin{aligned}
3L &\leq N-3 \\
\Rightarrow L &\leq \frac{N}{3}-1
\end{aligned} \tag{68}
$$

Thus output will be higher and prices lower as long as the number of firms in the leader group is less than 1/3 of the total firms in the industry. When L+1 is larger than N/3, prices will not fall and quantity supplied will be lower.

3. Vertical Mergers

3.1. Introduction to vertical mergers.

3.1.1. *Complementary products.* Two inputs (x_i, x_j) are said to be complementary in production if

$$\frac{\partial f\ (x)}{\partial x_i\,\partial x_j}\ >\ 0 \tag{69}$$

The idea is that using them together will increase the productivity of the process. Inputs are said to be technically competitive if

$$\frac{\partial f\ (x)}{\partial x_i\,\partial x_j}\ <\ 0 \tag{70}$$

These definitions hold all other inputs constant. More general definitions are based on the elasticity of substitution or on factor demands. For example we say that two inputs are complements if

$$\sigma_{ij}\ <\ 0 \tag{71}$$

where σ_{ij} is the elasticity of substitution (Allen, Hicks, or Morishima) between x_i and x_j. We also say that two inputs are gross complements if

$$\frac{\partial x_i\,(p,\ w)}{\partial w_j}\ =\ \frac{\partial x_i\ (w,\ y^*)}{\partial w_j}\ +\ \frac{\partial x_i\ (w,\ y^*)}{\partial y}\ \frac{\partial y\,(p,\ w)}{\partial w_j}\ \leq\ 0 \tag{72}$$

where x_i represents an ordinary demand curve. The idea is that there are economies to using the two inputs together.

3.1.2. *Problem of firms in a vertical chain.* Often firms in a vertical chain produce products that are complementary in the sense that the output of one firm is an input for another firm. In such cases the output of one firm will depend on the output of the other firm.

3.1.3. *Problem of two firms producing complementary products.* Sometimes consumers may prefer to buy products in fixed proportions. Examples include right and left handed gloves, right and left ski boots, oxygen and acetylene, computer chip pairs, etc. In such a situation the demand for one product is clearly related to the demand for the other product.

3.2. A simple monopoly problem. Assume that there are two goods which are used together in consumption. A classic example is nuts and bolts. Assume that each is produced by a monopolist. The nut monopolist has constant cost equal to c_N while the bolt monopolist has constant cost equal to c_B. A consumer who wants to buy 100 bolts also wants to buy 100 nuts so it is the combined price $p_B + p_N$ that is relevant. Let the demand curve for nut and bolt pairs be given by

$$Q\ =\ A\ -\ (p_B\ +\ p_N) \tag{73}$$

This is also the demand curve for the individual producers Because the two goods are always bought in pairs. Thus we obtain

$$\begin{aligned} Q_B\ &= A\ -\ (p_B\ +\ p_N) \\ Q_N\ &= A\ -\ (p_B\ +\ p_N) \end{aligned} \tag{74}$$

It is obvious that the price chosen by the other firm affects the demand for a given firm's product. Now write these demand curves in inverse form as

$$\begin{aligned} p_B &= A - p_N - Q_B \\ p_N &= A - p_B - Q_N \end{aligned} \tag{75}$$

We can then write total revenue and marginal revenue as

$$\begin{aligned} R_B &= (A - p_N - Q_B)\,Q_B \\ MR_B &= A - p_N - 2Q_B \\ R_N &= (A - p_B - Q_N)\,Q_N \\ MR_N &= A - p_B - 2Q_N \end{aligned} \tag{76}$$

If we now set these equal to marginal cost we obtain

$$\begin{aligned} MR_B &= A - p_N - 2Q_B = c_B \\ \Rightarrow Q_B &= \frac{A - p_N - c_B}{2} \\ MR_N &= A - p_B - 2Q_N = c_N \\ \Rightarrow Q_N &= \frac{A - p_N - c_N}{2} \end{aligned} \tag{77}$$

If we now substitute these quantities into the inverse demand curves we obtain

$$\begin{aligned} p_B &= A - p_N - Q_B \\ &= A - p_N - \left(\frac{A - p_N - c_B}{2}\right) \\ &= \left(\frac{A - p_N + c_B}{2}\right) \\ p_N &= A - p_B - Q_N \\ &= A - p_B - \left(\frac{A - p_B - c_N}{2}\right) \\ &= \left(\frac{A - p_B + c_N}{2}\right) \end{aligned} \tag{78}$$

The optimal price for each firm depends on the price charged by the other firm. Whatever price is being charged by the bolt maker will be obvious to the nut maker and vice versa. If the bolt maker lowers price, the nut maker will see an increase in demand and so forth. This market will be in equilibrium only when the two equations are simultaneously satisfied. We can find this point by substitution as follows

$$\begin{aligned}
p_B &= \left(\frac{A - p_N + c_B}{2}\right) \\
p_N &= \left(\frac{A - p_B + c_N}{2}\right) \\
\Rightarrow p_B &= \left(\frac{A - \left(\frac{A - p_B + c_N}{2}\right) + c_B}{2}\right) \\
&= \frac{A}{4} + \frac{p_B}{4} - \frac{c_N}{4} + \frac{c_B}{2} \\
\Rightarrow \frac{3}{4} p_B &= \frac{A - c_N + 2c_B}{4} \\
\Rightarrow p_B &= \frac{A - c_N + 2c_B}{3}
\end{aligned} \tag{79}$$

Similarly for p_N

$$p_N = \frac{A - c_B + 2c_N}{3} \tag{80}$$

The total price $p_N + p_B$ is given by the sum as is equal to

$$\begin{aligned}
p_N + p_B &= \frac{A - c_B + 2c_N}{3} + \frac{A - c_N + 2c_B}{3} \\
&= \frac{2A + c_B + c_N}{3}
\end{aligned} \tag{81}$$

The number of pairs sold is

$$\begin{aligned}
Q &= A - (p_B + p_N) \\
&= A - \frac{2A + c_B + c_N}{3} \\
&= \frac{A - c_B - c_N}{3}
\end{aligned} \tag{82}$$

Now compare this to the result if the two firms were merged and simply chose the total production of nuts and bolts. Inverse demand is given by

$$p_B + p_N = A - Q \tag{83}$$

Revenue and marginal revenue are given by

$$\begin{aligned}
Revenue &= (A - Q)\,Q \\
MR &= A - 2Q
\end{aligned} \tag{84}$$

The marginal cost of production is $c_B + c_N$. Equating marginal revenue and marginal costs will give

$$\begin{aligned}
MR &= A - 2Q = c_B + c_N \\
\Rightarrow Q &= \frac{A - c_b - c_n}{2}
\end{aligned} \tag{85}$$

This is clearly larger than in the case where the monopolists acted independently. Profits are lower with separate firms and prices to consumers are higher. The intuition behind the foregoing argument is straightforward. Each firm's pricing decision imposes an externality on the other firm. For example, a high price for computer hardware will reduce demand for PC's. It will also reduce demand for programs and operating systems. The first effect is taken into account by the hardware manufacturer. The second is not. The same is true, of course, in reverse. The software manufacturer does not take into account the impact its price choice has on the demand for hardware. In the noncooperative equilibrium, the price of both goods will be too high. If say, the hardware firm were to cuts its price, this would generate additional demand and additional profit for the software firm. However, because the hardware firm does not receive any of this additional profit, it does not reduce its price. This implies that, with cooperation, both firms would lower their price and be better off. Consumers, too, would gain as a result of lower prices and expanded output. One way to achieve the profit and efficiency gains of cooperation is for the two firms to merge. Such a merger creates a singly entry and, therefore, permits the externality to be internalized. The combined hardware and software firm will certainly wish to maximize its total profit which means that it will wish to price the two individual goods so as to maximize the joint profit of each. Thus, whenever firms with monopoly power produce complementary products, they have a strong incentive either to merge or to devise some other method to assure cooperative production and pricing of the complementary products.

3.3. **Vertical Mergers as a Way to Resolve a Common Type of Complementarity.**

3.3.1. *The incentive behind vertical mergers.* Vertical mergers involve combination of firms at different stages of the production stream. The convention is to label those firms farthest from the final consumer of the product as *upstream* and those closest to that consumer as *downstream*. Film companies and movie theaters are an example. In this case, the film company is the upstream firm and the theater that rents the film and then uses it as an input to provide its customers with a cinematic experience is the downstream firm. Manufacturers and retailers have a similar upstream-downstream relation. The point to understand is that all such relationships can be usefully viewed through the lens of complementarity. The implication is that vertical relationships between two firms—each with monopoly power—will lead to suboptimal pricing and economic inefficiency in the absence of some mechanism to coordinate the decisions of the two firms. In the case of vertically-related firms, this issue is typically referred to as the problem of *double marginalization*.

3.3.2. *Set-up of vertical merger problem.*

1. Firms
   There are two firms: a single upstream supplier, the manufacturer, who sells a unique product to a single downstream firm, the retailer.
2. Prices and costs
   The manufacturer produces the good at constant unit cost c, and sells it to the retailer at a wholesale price r. The retailer resells the product to consumers at price p. The retailer has no retailing cost.
3. Demand Consumer demand for the good is described by a linear demand function,

$$p = A - BQ, \text{ with } c < A. \tag{86}$$

4. Profit
   Because the retailer buys a unit of the good at whole price r and simply resells it at price p, the retailer's profit can be written as:

$$\Pi^D(p,\, r) = (p - r)Q = (A - BQ)\,Q - rQ \tag{87}$$

3.3.3. *Profit maximization for retailer.* Profit will be maximized by choosing an output at which the retailer's marginal revenue is equal to its cost. Carrying out the optimization will give

$$\begin{aligned} \frac{d\,\Pi^D(p,\, r)}{dQ} &= \frac{d\,[\,(A\,-\,BQ)\,Q\,-\,rQ\,]}{dQ} = 0 \\ &\Rightarrow\; A\,-\,2BQ\,-\,r\,=0 \\ &\Rightarrow\; Q\,=\frac{A\,-\,r}{2B} \end{aligned} \tag{88}$$

If we substitute this into the demand equation we find the equilibrium price of

$$\begin{aligned} p\, &= A\,-\,BQ \\ &= A\,-\,B\,\left[\frac{A\,-\,r}{2B}\right] \\ &= \left[\frac{A\,+\,r}{2}\right] \end{aligned} \tag{89}$$

Profit is given by

$$\begin{aligned} \Pi^D(p,\, r)\, &= (p-\, r)Q\, = \left(\frac{A\,+\,r}{2}\,-\,r\right)Q \\ &= \left(\frac{A\,-\,r}{2}\right)\left[\frac{A\,-\,r}{2B}\right] \\ &= \frac{(A\,-\,r)^2}{4B} \end{aligned} \tag{90}$$

We can represent this graphically as in figure 1

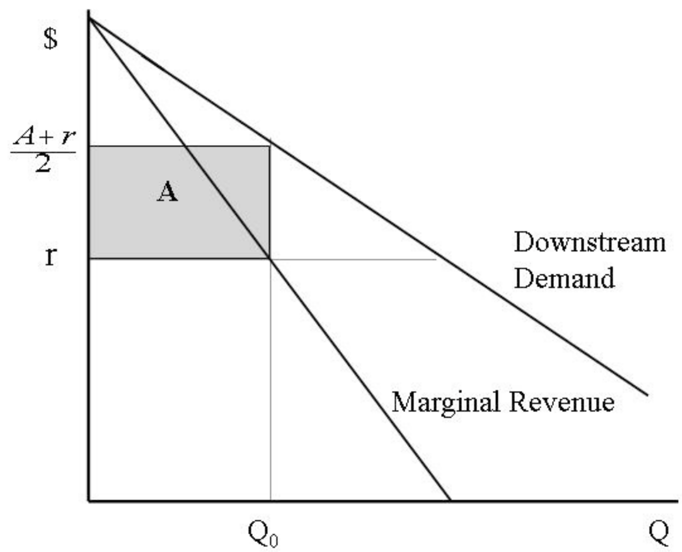
The shaded area is the profit for this retailer. The marginal revenue curve is also the demand curve for the product of the manufacturer. For different values of r, different amounts of the input are demanded. This can be made more formal by noting that quantity sold in the final products market is also the quantity that must be supplied by the manufacturer. Specifically the quantity $Q\;=\;\frac{(A\,-\,r)}{2B}$ is the amount of the product that the manufacturer will sell to the retailer at price r. We can also solve this equation for the inverse demand curve as

$$r\;=\;A\;-\;2\,BQ^U$$

where Q^U denotes the quantity supplied by the wholesaler. In short, *the inverse demand facing the upstream manufacturer at wholesale price r, namely, r = A - 2BQ^U, is also the marginal revenue function facing the retailer.*

3.3.4. *Profit maximization for the wholesaler.* Because the marginal revenue curve of the retailer and the inverse demand function confronting the upstream firm are the same, we can formally write the latter as: r = A - 2B Q^U. This allows us to derive the profit-maximizing price that the upstream firm will set for its product. The optimal level of Q^U is derived as follows

FIGURE 1. Independent Retailer's Optimal Price



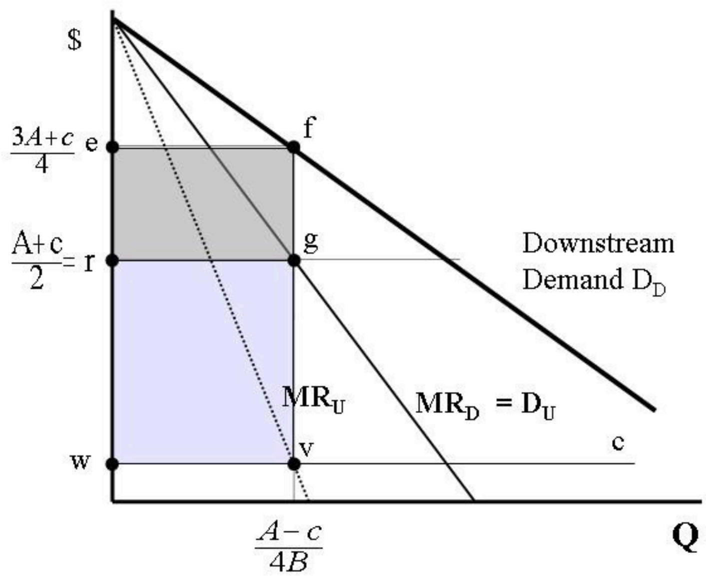
$$\begin{aligned}
\Pi^U(c,\ r) &= (A \ - \ 2BQ^U)\,Q^U \ - \ c\,Q^U \\
\frac{d\,\Pi^U(c,\ r)}{dQ^U} &= \frac{d\,\left[\,(A \ - \ 2BQ^U)\,Q^U \ - \ cQ^U\,\right]}{dQ^U} \ = \ 0 \\
\Rightarrow \ A \ - \ 4BQ^U \ - \ c &= 0 \qquad (91) \\
\Rightarrow \ Q^U &= \frac{A \ - \ c}{4B} \\
\Rightarrow \ r^U &= \frac{A \ + \ c}{2}
\end{aligned}$$

The interaction between the independent manufacturer and the independent retailer, both trying to maximize their individual profit, is shown in figure 2. When the manufacturer sets the price r^U = (A + c)/2, the retailer sets the price p^D = (3A + c)/4 as can be seen from

$$\begin{aligned}
p^D &= \left[\frac{A \ + \ r}{2}\right] \\
&= \frac{A}{2} \ + \ \frac{\frac{A \ + \ c}{2}}{2} \qquad (92) \\
&= \frac{3A \ + \ c}{4}
\end{aligned}$$

The retailer then sells Q^D = (A - c)/4B. This is, of course, precisely the amount the manufacturer anticipated it would sell when it set its upstream price, r^U = (A + c)/2, in the first place. The combined

FIGURE 2. Independent Retailer's Optimal Price when r = $\frac{A-c}{2}$



profits of the manufacturer and retailer are shown in the figure as areas refg and wrgv, respectively. The manufacturer's profit at this optimal price and output is equal to: Π^U = (A - c)^2/8B.

3.3.5. *Merging the firms.* Consider now what happens if the two firms merge so that the manufacturer is no longer independent, but merely the upstream division of the integrated firm, supplying the good to the downstream retail division of the same parent company. The good is still produced at constant marginal c. The only strategic question facing this merged firm is: what price should it charge consumers at the retail level? This effectively transforms the integrated firm into a simple monopoly whose goal is to maximize monopoly profit. This profit is given by:

$$\pi^I(p,\ c) = (p - c)Q = (A - BQ)Q - cQ \tag{93}$$

Maximizing profit will yield

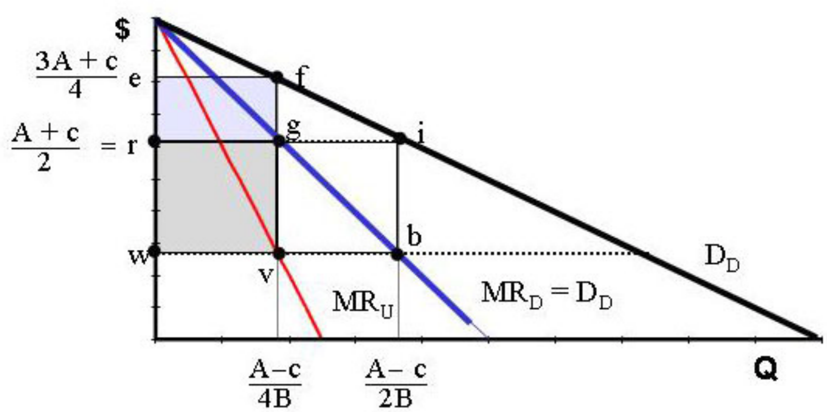
$$\begin{aligned} \frac{d\,\Pi^I(p,\ c)}{dQ^I} &= \frac{d\left[(A - BQ^I)Q^I - cQ^I\right]}{dQ} = 0 \\ \Rightarrow A - 2BQ^I - c &= 0 \\ \Rightarrow Q^I &= \frac{A - c}{2B} \\ \Rightarrow p^I &= \frac{A + c}{2} \end{aligned} \tag{94}$$

Comparing this to equation 92, we see that the integrated firm's optimal retail price to consumers is *lower* than the price facing those consumers when set by an independent retailing firm. Correspondingly, the integrated firm ultimately sells more of the product than do the two, non-integrated firms. Moreover, the integrated firm also earns *more* profit than the separate manufacturer and the retailer combined. The profit earned by the integrated firm is

$$\begin{aligned} \pi^I(p,\ c) &= (p\ -\ c)\,Q \\ &= \left[\frac{A\ +\ c}{2}\ -\ c\right]\frac{A\ -\ c}{2B} \\ &= \left[\frac{A\ -\ c}{2}\right]\frac{A\ -\ c}{2B} \\ &= \left[\frac{(A\ -\ c)^2}{4\,B}\right] \end{aligned} \tag{95}$$

This is 12.5 percent greater than the profit of the upstream manufacturer $\Pi^U = (A - c)^2/8B$ plus the profit of the retailer $\Pi^D = (A - c)^2/16B$, which is equal to $\Pi^U + \Pi^D = 3(A - c)^2/16B$. From a social welfare point of view, integrating the two monopoly firms has benefitted everyone. Total profit is increased, but consumer surplus is also raised as more of the good is sold at a lower price. The gains from integration are illustrated in figure 3. What was the retailer's profit, area refg, has now been redistributed to consumers as surplus. In addition, consumers also gain the area fgi. The manufacturer's profit has doubled from area wrgv to wrgb and this more than offsets the loss of the retailer's profit.

FIGURE 3. Market Equilibrium with Merger



We can now see the clear parallel between this case and that of two monopoly firms selling complementary goods. In both cases, integration brings an efficiency gain and extra profit because it makes it possible for the separate activities to be coordinated and, thereby, to internalize the externality each imposes on the other. In the absence of such a merger, the final price will reflect a double marginalization. The independent manufacturer will mark up its price to the retailer who then compounds that price-cost distortion by adding a further markup in setting a price to the consumer. This is the source of the old saying: "What is worse than a monopoly? A chain of monopolies".

One further point should be stressed in reviewing the above analysis. This is that the gains brought on by integration hinge crucially on the fact that the pre-merger setting was one of the monopoly at **both** levels of activity, manufacture and retail. If, instead, we had started with either a competitive manufacturing sector upstream selling to a monopoly downstream, or a monopoly stream selling to a competitive retail sector, there would be no efficiency gain to vertical integration. Price competition upstream among manufacturers leads to a wholesale price equal to marginal cost. Similarly, competition among retailers downstream brings the retail price equal to r. In either case, the margin of price over cost will be set to zero at one of the two levels of activity. Hence, in these settings, no *double* marginalization can occur.

We should also note one additional qualification to the foregoing analysis. The benefits that the vertical merger of an upstream and downstream monopolist assumes that the downstream firm operates under conditions of fixed input proportions. This means we have assumed that the downstream firm uses some fixed amount of the upstream firm's product for every unit of output that the downstream firm sells. In our example of a producer and a downstream retailer this assumption may make sense. The retailer has to have one unit of the manufacturer's product for every unit it resells to its customers. Yet for other situations this assumption is too strong. For example, if the upstream firm is a steel producer and the downstream firm is an automobile manufacturer, the steel firm's decision to charge the car company a price, r, that includes a high markup, may induce the auto maker to switch to aluminum or perhaps fiberglass, both assumed to be competitively supplied. In such a case, the benefits of the car company integrating backwards into the steel market are less clear-cut. If substitution possibilities are great enough, they may be sufficient to offset the benefits that accrue from removal of double marginalization.

3.4. **Vertical Merger to Facilitate Market Foreclosure.**

3.4.1. *Why market foreclosure may be important.* One motive for vertical integration is foreclosure, that is, the merger of vertically related firms so that an upstream-downstream company that can either deny downstream rivals a source of inputs, or upstream competitors a market for their products.

Consider, for example, a situation in which a motor car manufacturer decides to merge with the company that supplies its gear boxes. This is likely to have two effects. First, other gear-box manufacturers will no longer be able to compete for this auto maker's business. Secondly, it is possible that the merged firm will prefer not to sell the gear-boxes, which as a result of the merger, it now produces, to any other auto company. The merger firm may also attempt a "price squeeze" by offering to sell its gearboxes to outside firms but only at exorbitant prices.[1]

Such *market foreclosure* effects of vertical mergers may so reduce competitive forces in both the upstream and downstream markets as to totally offset the benefits from any mitigation of the double marginalization problem. Indeed, this is probably one of the major reasons why regulatory agencies investigate the effects of vertical mergers. An early Alcoa case was one particularly famous instance of foreclosure. Alcoa was accused of maintaining a powerful market position by extracting promises from power companies not to supply electricity—vital to the refining of aluminum—to competing aluminum producers. In addition, they were accused of employing a price squeeze by charging very high prices for aluminum ingots that were used by companies that competed with Alcoa in certain downstream markets, such as the aluminum sheet market.

3.4.2. *Set-up of a simple foreclosure model.* Assume that there is an upstream market containing n_U firms and a downstream market containing n_D firms. Of these firms, n are vertically integrated, so that there are n_U - n independent, non-integrated upstream suppliers and n_D - n independent, non-integrated downstream firms. Each of the upstream units incurs a constant marginal cost c_U in making the upstream product, exactly one unit of which is needed to make one unit of the downstream product. Downstream production incurs additional marginal costs c_D per unit. The products of the downstream firms are identical and both upstream and downstream firms act as Cournot competitors. Demand for the products of the downstream firm is:

$$p^D = A - BQ^D \tag{96}$$

where Q^D is aggregate output of the downstream product.

3.4.3. *Profit for the three types of firms.* Profit for an integrated downstream firm is:

$$\pi_{D_i} = (p^D - c_U - c_D)\, q_{D_i} = (A - c_U - c_D - BQ_D)\, q_{D_i} \tag{97}$$

Profit for a non-integrated downstream firm is:

$$\pi_{D_n} = (p^D - p^U - c_D)\, q_{D_n} = (A - p^U - c_D - B\,Q_D)\, q_{D_n} \tag{98}$$

Profit of a non-integrated upstream firm is:

$$\pi_{U_n} = (p^U - c_U)\, q_{U_n} \tag{99}$$

[1]There are, in fact, a whole host of methods by which merged firms can effectively foreclose their competitors. For an illustration of some of these, see Krattenmaker, T., and S. Salop (1986) "Anticompetitive exclusion: raising rivals' costs to achieve power over price," *Yale Law Journal*, pp. 209-295.

where p^u is the price charged by each non-integrated upstream firm for its product. These equations indicate an immediate advantage that the integrated firms enjoy: They obtain their inputs at marginal cost while the non-integrated firms must pay a higher price p^u.

3.4.4. *Implications of the model structure.*

i. The equations show that the integrated firms will not purchase their inputs from non-integrated upstream suppliers. This is obvious because for any non-integrated upstream firms to stay in business it is necessary that the market price cover its cost, $p^u > c_U$. However, if this is the case, then a downstream division of an integrated firm will find it cheaper to obtain its inputs from its upstream affiliate at cost rather than buy them on the open market for price, p^u.
ii. The equations can also be used to show that the integrated firms will not supply the upstream product to non-integrated downstream buyers. This second result is more complex and requires more analysis.

3.4.5. *Integrated firms will not supply the upstream product to non-integrated downstream buyers.* For the non-integrated downstream firms to stay in business it is necessary that p^d - p^u - $c_D > 0$, i.e., that the downstream price at least covers the non-integrated firms' production cost, including the cost of upstream supplies. If, in equilibrium, the upstream division of an integrated firm is selling some of its output to a non-integrated downstream firm it will earn a margin of p^u - c_U on each unit sold. If, instead, it takes this output off the outside market and uses it itself, total output of the final product will be unaffected and so the downstream price will not change. But this means that the integrated firm will earn p^d - c_U - c_D on every unit so diverted. So, if the condition for a non-integrated downstream firm to exist, p^d - p^u - $c_D > 0$, is met, the margin earned on units diverted to the integrated firm's downstream affiliate exceeds what it earns selling to the independent downstream firm. As a result, the integrated firms will not sell their intermediate products to non-integrated firms. In short, foreclosure happens. However, the fact that foreclosure happens is not the same as saying that such foreclosure is harmful to consumers. To determine this, we need to identify the effect of vertical integration and market foreclosure on the final market price.

3.4.6. *Impact of foreclosure on consumers.* Consider first the upstream market. Vertical integration and the market foreclosure just described reduces the number of independent competing firms and reduces their number of customers (the remaining independent downstream companies). Both factors reduce competition in the market in which the upstream product is actually sold (rather than transformed internally between affiliates). On the other hand, these independent upstream firms are supplying downstream companies who are at a cost disadvantage relative to their integrated downstream competitors (they pay the price p_u rather than the fee, c_U). This constrains the ability of the independent downstream firms to raise the price to consumers. If there are "enough" independent upstream firms following a vertical merger, the anticompetitive effects of market foreclosure by having another integrated firm will be relatively minor, and should be offset by the cost-reducing effects of vertical integration. In this case, such integration will reduce the final product price and so will not necessarily be harmful to consumers. To put it another way, the regulatory authorities can be reasonably assured that a vertical merger and the market foreclosure effects it has will not have detrimental effects on consumers, if there remains a reasonably larger sector of independent upstream suppliers.

This analysis does not take into account the potential strategic reactions of downstream (and upstream) firms whose markets are threatened by vertical mergers. Is it likely that they will stand passively by and see their markets destroyed? One obvious reaction of a downstream firm to the threat of foreclosure by a vertically integrated rival is for this firm to seek a merger with an upstream supplier. This will have the effect of significantly increasing competitive pressures in the downstream market because the downstream firms' costs are lower, particularly if firms compete in prices rather than quantities.