

NONCOOPERATIVE OLIGOPOLY MODELS

1. INTRODUCTION AND DEFINITIONS

Definition 1 (Oligopoly). Noncooperative oligopoly is a market where a small number of firms act independently but are aware of each other's actions.

1.1. Typical assumptions for oligopolistic markets.

1. Consumers are price takers.
2. All firms produce homogeneous products.
3. There is no entry into the industry.
4. Firms collectively have market power: they can set price above marginal cost.
5. Each firm sets only its price or output (not other variables such as advertising).

1.2. Summary of results on oligopolistic markets.

1. The equilibrium price lies between that of monopoly and perfect competition.
2. Firms maximize profits based on their beliefs about actions of other firms.
3. The firm's expected profits are maximized when expected marginal revenue equals marginal cost.
4. Marginal revenue for a firm depends on its residual demand curve (market demand minus the output supplied by other firms)

2. OLIGOPOLY MODELS AND GAME THEORY

Definition 2 (Game Theory). A *game* is a formal representation of a situation in which a number of decision makers (players) interact in a setting of **strategic interdependence**. By that, we mean that the welfare of each decision maker depends not only on her own actions, but also on the actions of the other players. Moreover, the actions that are best for her to take may depend on what she expects the other players to do. We say that game theory analyzes interactions between rational, decision-making individuals who may not be able to predict fully the outcomes of their actions.

Definition 3 (Nash equilibrium). A set of strategies is called a *Nash equilibrium* if, holding the strategies of all other players constant, no player can obtain a higher payoff by choosing a different strategy. In a Nash equilibrium, no player wants to change its strategy.

2.1. Behavior of firms in oligopolistic games.

1. Firms are rational.
2. Firms reason strategically.

2.2. Elements of typical oligopolistic games.

1. There are two or more firms (not a monopoly).
2. The number of firms is small enough that the output of an individual firm has a measurable impact on price (not perfect competition). We say each firm has a few, but only a few, rivals.
3. Each firm attempts to maximize its expected profit (payoff).
4. Each firm is aware that other firm's actions can affect its profit.
5. Equilibrium payoffs are determined by the number of firms, the rules of the games and the length of the game.

2.3. Comparison of oligopoly with competition. In competition each firm does not take into account the actions of other firms. In effect they are playing a game against an impersonal market mechanism that gives them a price that is independent of their own actions. In oligopoly, each firm explicitly takes account of other firm's expected actions in making decisions.

2.4. Single period models.

Definition 4 (Single period or static games). Firms or players "meet only once" in a single period model. The market then clears one and for all. There is no repetition of the interaction and hence, no opportunity for the firms to learn about each other over time. Such models are appropriate for markets that last only a brief period of time.

There are three main types of static oligopoly models.

1. Cournot
2. Bertrand
3. Stackelberg

3. THE COURNOT MODEL

3.1. Historical background. Augustin Cournot was a French mathematician. His original model was published in 1838. He started with a duopoly model. His idea was that there was one incumbent firm producing at constant unit cost of production and there was one rival firm considering entering the market. Since the incumbent was a monopolist, $p > MC$, and there was potential for a rival to enter and to make a profit. Cournot postulated that the rival firm would take into account the output of the incumbent in choosing a level of production. Similarly he postulated that the monopolist would consider the potential output of the rival in choosing output.

3.2. The two-firm Cournot duopoly model and its solution.

3.2.1. General assumptions of the Cournot model.

1. Two firms with no additional entry.
2. Homogeneous product such that $q_1 + q_2 = Q$ where Q is industry output and q_i is the output of the i^{th} firm.
3. Single period of production and sales (consider a perishable crop such as cantaloupe or zucchini).
4. Market and inverse market demand is a linear function of price. We write this linear inverse demand as follows

$$\begin{aligned}
 p &= A - BQ = A - B(q_1 + q_2) \\
 &= A - Bq_1 - Bq_2 \\
 \Rightarrow Q &= \frac{A}{B} - \frac{p}{B} = \frac{A - p}{B}
 \end{aligned} \tag{1}$$

5. Each firm has constant and equal marginal cost equal to c . With constant marginal cost, average cost is also constant and equal to c .
6. The decision variable in the quantity of output to produce and market.

3.2.2. *Example Model.* Assume market demand is given by

$$Q(p) = 1000 - 1000p \quad (2)$$

This implies that inverse demand is given by

$$\begin{aligned} Q(p) &= 1000 - 1000p \\ \Rightarrow 1000p &= 1000 - Q \\ \Rightarrow p &= 1 - 0.001Q \\ &= 1 - .001(q_1 + q_2) \\ &= 1 - 0.001q_1 - 0.001q_2. \end{aligned} \quad (3)$$

With this demand function, when $p = \$1.00$, $Q = 0$ and when $p = 0$, $Q = 1000$. Also notice that for $p = A - BQ$ to have a positive price with $A > 0$ and $B > 0$ it must be that $\frac{A}{B} > Q$.

For this example, assume that

$$AC = MC = \$0.28. \quad (4)$$

3.2.3. *Residual demand curves in oligopoly models.* Assume that there are two firms: Firm 1 and Firm 2. If Firm 2 believes that Firm 1 will sell q_1 units of output, then its residual inverse demand curve is given by

$$p = A - Bq_1 - Bq_2 = (A - Bq_1) - Bq_2 \quad (5)$$

In equation 5 we view $(A - Bq_1)$ as a constant. We can also write this in quantity dependent form as

$$q_2(p) = Q(p) - q_1 \quad (6)$$

We obtain equation 6 by shifting the market demand curve to the left by q_1 units. For example if $q_1 = 300$ then the residual demand curve will hit the horizontal axis at 700 units. Thus

$$\begin{aligned} q_2(p) &= Q(p) - q_1 \\ &= (1000 - 1000p) - 300, \quad q_1 = 300 \\ &= 700 - 1000p \end{aligned} \quad (7)$$

Now if $p = 0$ then $q_2(p) = 700$. The residual inverse demand curve is given by substituting 300 in equation 7 to obtain

$$\begin{aligned} p &= 1 - 0.001q_1 - 0.001q_2 \\ &= 1 - (0.001)(300) - 0.001q_2 \\ &= 0.70 - .001q_2. \end{aligned} \quad (8)$$

We find residual marginal revenue by taking the derivative of the residual revenue function. With residual demand given by $p = A - Bq_1 - Bq_2$, revenue for the second firm is given by

$$\begin{aligned} R_2 &= (A - Bq_1 - Bq_2)q_2 \\ &= Aq_2 - Bq_1q_2 - Bq_2^2 \end{aligned} \quad (9)$$

while residual marginal revenue is given by the derivative of residual revenue

$$\begin{aligned} R_2 &= Aq_2 - Bq_1q_2 - Bq_2^2 \\ MR_2 &= A - Bq_1 - 2Bq_2 \end{aligned} \quad (10)$$

For the example with residual demand given by $p = .70 - .001q_2$, revenue for the second firm is given by $R_2 = (.70 - .001q_2)q_2 = .70q_2 - .001q_2^2$. Residual marginal revenue is given by the derivative of residual revenue

$$\begin{aligned} R_2 &= .70q_2 - .001q_2^2 \\ \Rightarrow MR_2 &= \frac{dR_2}{dq_2} = .70 - .002q_2 \end{aligned} \quad (11)$$

3.2.4. *Profit maximization given a residual demand curve.* Setting marginal revenue equal to the constant marginal cost will allow us to solve for q_2 as a function of q_1 , c , and the demand parameters as follows

$$\begin{aligned} MR_2 &= \frac{dR_2}{dq_2} = A - Bq_1 - 2Bq_2 = MC_2 = c \\ \Rightarrow A - Bq_1 - 2Bq_2 &= MC_2 = c \\ \Rightarrow 2Bq_2 &= A - Bq_1 - c \\ \Rightarrow q_2 &= \frac{A - Bq_1 - c}{2B} \\ \Rightarrow q_2^* &= \frac{A - c}{2B} - \frac{q_1}{2} \end{aligned} \quad (12)$$

This is called the reaction function for Firm 2. For any level of Firm 1 output, it gives the optimal level of output for Firm 2. For the example firm, marginal cost is given by \$0.28, so setting marginal revenue equal to marginal cost with $q_1 = 300$ will give

$$\begin{aligned} MR_2 &= \frac{dR_2}{dq_2} = .70 - .002q_2 = MC_2 = .28 \\ \Rightarrow .70 - .002q_2 &= MC_2 = .28 \\ \Rightarrow .002q_2 &= .42 \\ \Rightarrow q_2^* &= \frac{.42}{.002} = 210 \end{aligned} \quad (13)$$

Rather than specifying a level of q_1 in this example we can perform the same analysis for a generic value of q_1 as in equation 9 or 10 as follows. In this case

$$\begin{aligned}
q_2(p) &= 1000 - 1000p - q_1 \\
&= 1000 - q_1 - 1000p \\
\Rightarrow p &= \frac{1000 - q_2 - q_1}{1000} \\
&= 1 - .001(q_1 + q_2) \\
\Rightarrow R &= [1 - .001(q_1 + q_2)]q_2 \\
&= q_2 - .001q_2^2 - .001q_1q_2 \\
\Rightarrow MR_2 &= \frac{dR_2}{dq_2} = 1 - .001q_1 - .002q_2
\end{aligned} \tag{14}$$

Setting marginal revenue equal to marginal cost we obtain

$$\begin{aligned}
MR_2 &= \frac{dR_2}{dq_2} = 1 - 0.001q_1 - 0.002q_2 = c \\
\Rightarrow 1 - 0.001q_1 - c &= 0.002q_2 \\
\Rightarrow q_2^* &= \frac{1 - 0.001q_1 - c}{.002} \\
&= \frac{1 - c}{.002} - 0.5q_1
\end{aligned} \tag{15}$$

For the example with marginal cost equal to \$0.28 we obtain

$$\begin{aligned}
MR_2 &= \frac{dR_2}{dq_2} = 1 - 0.001q_1 - 0.002q_2 = MC = .28 \\
\Rightarrow 0.002q_2 &= .72 - .001q_1 \\
\Rightarrow q_2^* &= \frac{0.72}{0.002} - .5q_1 \\
&= 360 - 0.5q_1
\end{aligned} \tag{16}$$

For different values of q_1 we get different values of q_2^* as in table 1.

Graphically we can depict the optimum for Firm 2 when it assumes Firm 1 produces 300 as in figure 1. We can also plot the reaction curve for Firm 2 given various levels of q_1 as in figure 2. When $q_1 = 300$ the optimum quantity of q_2 is at 210. The firm will attempt to charge a price of $p = 1 - .001(210+300) = .49$. Plugging this in the demand function will give $Q = 1000 - 1000(.49) = 510$.

Alternatively if Firm 2 acted competitively, equilibrium would be where the residual demand curve intersected marginal cost. This is at an output of 420. With q_1 at 300 this gives an industry output of 720. This then gives a market price of $p = 1 - .001(420+300) = .28$ which is equal to marginal cost.

Firm 1 is symmetric to Firm 2. Consider the profit maximization problem for Firm 1.

TABLE 1. Optimal Response of Firm 2 Given Output Level of Firm 1

q_1	q_2^*	$MR(q_2^*)$	MC
0.00	360.00	0.28	0.28
20.00	350.00	0.28	0.28
40.00	340.00	0.28	0.28
60.00	330.00	0.28	0.28
80.00	320.00	0.28	0.28
100.00	310.00	0.28	0.28
120.00	300.00	0.28	0.28
140.00	290.00	0.28	0.28
160.00	280.00	0.28	0.28
180.00	270.00	0.28	0.28
200.00	260.00	0.28	0.28
210.00	255.00	0.28	0.28
220.00	250.00	0.28	0.28
240.00	240.00	0.28	0.28
250.00	235.00	0.28	0.28
255.00	232.50	0.28	0.28
260.00	230.00	0.28	0.28
280.00	220.00	0.28	0.28
300.00	210.00	0.28	0.28
320.00	200.00	0.28	0.28
340.00	190.00	0.28	0.28
360.00	180.00	0.28	0.28
380.00	170.00	0.28	0.28
400.00	160.00	0.28	0.28
420.00	150.00	0.28	0.28
440.00	140.00	0.28	0.28
460.00	130.00	0.28	0.28
480.00	120.00	0.28	0.28
500.00	110.00	0.28	0.28
520.00	100.00	0.28	0.28
540.00	90.00	0.28	0.28
560.00	80.00	0.28	0.28
580.00	70.00	0.28	0.28
600.00	60.00	0.28	0.28
620.00	50.00	0.28	0.28
640.00	40.00	0.28	0.28
660.00	30.00	0.28	0.28
680.00	20.00	0.28	0.28
700.00	10.00	0.28	0.28
720.00	0.00	0.28	0.28

$$\begin{aligned}
\pi_1 &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} [(A - B q_1 - B q_2) q_1 - c q_1] \\
&= \max_{q_1} [A q_1 - B q_1^2 - B q_1 q_2 - c q_1] \\
\Rightarrow \frac{d\pi_1}{dq_1} &= A - 2 B q_1 - B q_2 - c = 0 \\
\Rightarrow 2 B q_1 &= A - B q_2 - c \\
\Rightarrow q_1^* &= \frac{A - c}{2 B} - \frac{q_2}{2}
\end{aligned} \tag{17}$$

FIGURE 1. Optimum for Firm 2 when Firm 1 produces 300

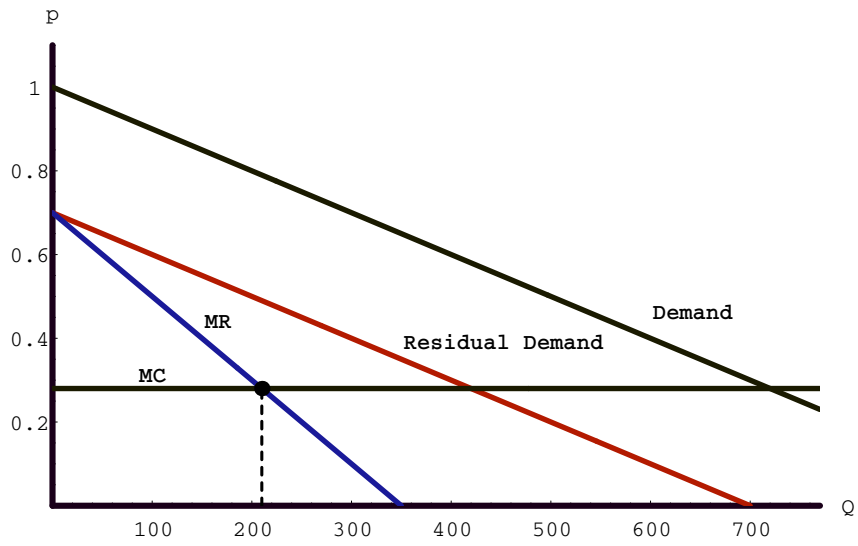
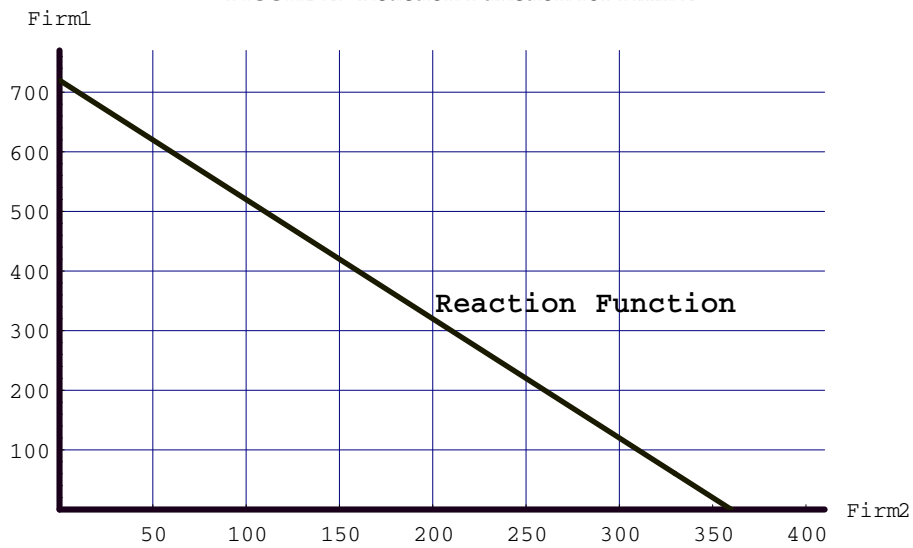


FIGURE 2. Reaction Function for Firm 2



Second order conditions require that $-2Bq_1 < 0$ which implies that $B > 0$. Equation 17 gives the reaction function for Firm 1. The parameters in the example problem are $A = 1$, $B = .001$ and $c = .28$. This then gives a response function of

$$\begin{aligned}
 q_1^* &= \frac{A - c}{2B} - \frac{q_2}{2} \\
 &= \frac{1 - .28}{(2)(.001)} - 0.5 q_2 \\
 &= \frac{.72}{.002} - 0.5 q_2 \\
 &= 360 - 0.5 q_2
 \end{aligned} \tag{18}$$

Now suppose Firm 1 thinks that Firm 2 will produce 210 units. Firm 1 will then produce

$$\begin{aligned}
 q_1^* &= 360 - 0.5 q_2 \\
 &= 360 - (0.5)(210) \\
 &= 360 - 105 \\
 &= 255
 \end{aligned} \tag{19}$$

Unfortunately when Firm 1 thinks Firm 2 will produce 210, it chooses to produce 255, not the 300 that would have induced Firm 2 to produce 210. Thus this set of beliefs and decisions is not a Nash equilibrium since Firm 2 would now prefer to produce not 210, but 232.5. But if Firm 2 produces 232.5 then Firm 1 may also choose a different level of output etc.

3.2.5. *Finding a Nash equilibrium in the Cournot Model.* In order to find an equilibrium we need to find a point where each firm's beliefs about the other firm are fulfilled. This can be done using the reaction functions for each firm (which are the same). Consider first the best response for Firm 1 if Firm 2 produces nothing. This is given by

$$\begin{aligned}
 q_1^* &= \frac{A - c}{2B} - \frac{0}{2} \\
 &= \frac{A - c}{2B}
 \end{aligned} \tag{20}$$

which is just the monopoly solution. But what if Firm 1 did produce $\frac{A-c}{2B}$. The best response of Firm 2 would be to produce

$$\begin{aligned}
 q_2^* &= \frac{A - c}{2B} - \frac{q_1}{2} \\
 &= \frac{A - c}{2B} - (0.5) \left(\frac{A - c}{2B} \right) \\
 &= \frac{A - c}{4B}
 \end{aligned} \tag{21}$$

But then Firm 1 will no longer assume it is a monopolist but will take this output as given and produce

$$\begin{aligned}
 q_1^* &= \frac{A - c}{2B} - \frac{q_2}{2} \\
 &= \frac{A - c}{2B} - (0.5) \left(\frac{A - c}{4B} \right) \\
 &= \frac{3(A - c)}{8B}
 \end{aligned} \tag{22}$$

Cournot's insight was that for an outcome to be an equilibrium, it must be the case that each firm is responding optimally to the (optimal) choice of its rival. Each firm must choose a best response based upon

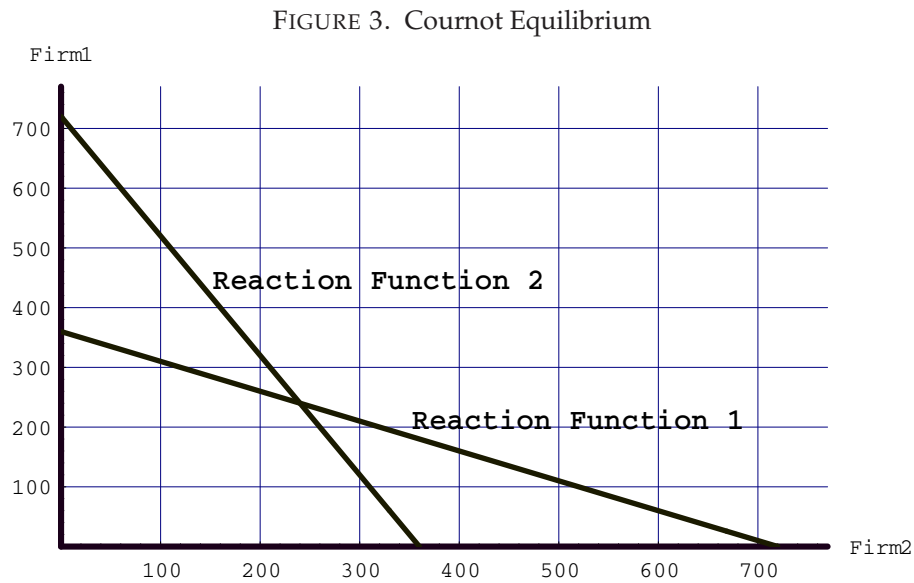
a prediction about what the other firm will produce, and in equilibrium, each firm's prediction must be correct. An equilibrium requires that both firms be on their respective reaction curves. To find this point we can substitute the optimal response for Firm 1 into the response function for Firm 2 as follows

$$\begin{aligned}
 q_2^* &= \frac{A - c}{2B} - \frac{q_1}{2} \\
 &= \frac{A - c}{2B} - (0.5) \left(\frac{A - c}{2B} - \frac{q_2}{2} \right) \\
 &= \frac{A - c}{4B} + \frac{q_2}{4} \\
 \Rightarrow \frac{3q_2^*}{4} &= \frac{A - c}{4B} \\
 \Rightarrow q_2^* &= \frac{A - c}{3B}
 \end{aligned} \tag{23}$$

Similarly we find that

$$q_1^* = \frac{A - c}{3B} \tag{24}$$

We can also see this graphically by plotting the reaction functions as in figure 3



For the numerical example we obtain the optimal output of Firm 1 as $q_1^* = \frac{1 - .28}{.003} = 240$. We then get the optimal output of Firm 2 as

$$\begin{aligned}
 q_2 &= 360 - .5q_1 \\
 &= 360 - .5(240) \\
 &= 360 - 120 \\
 &= 240
 \end{aligned} \tag{25}$$

Industry output is the sum of the output of the two firms

$$\begin{aligned}
 Q^* &= q_1^* + q_2^* \\
 &= \frac{A - c}{3B} + \frac{A - c}{3B} \\
 &= \frac{2(A - c)}{3B} \\
 &= 480
 \end{aligned} \tag{26}$$

The Cournot price is given by

$$\begin{aligned}
 p &= (A - BQ^*) \\
 &= \left(A - B \left(\frac{2(A - c)}{3B} \right) \right) \\
 &= \left(A - \frac{2}{3}A + \frac{2}{3}c \right) \\
 &= \left(\frac{1}{3}A + \frac{2}{3}c \right) \\
 &= \frac{A + 2c}{3} \\
 &= .52
 \end{aligned} \tag{27}$$

Each firm then has profit of

$$\begin{aligned}
 \pi_i &= p_i^* q_i^* - c q_i^* \\
 &= \left(\frac{1}{3}A + \frac{2}{3}c \right) q_i^* - c q_i^* \\
 &= \left(\frac{1}{3}A - \frac{1}{3}c \right) q_i^* \\
 &= \frac{1}{3} (1 - .28) (240) \\
 &= 57.6
 \end{aligned} \tag{28}$$

Thus the Cournot equilibrium for this market is $q_1^* = q_2^* = 240$. Each firm believes the other will sell 240 units, each will sell 240 units. At this point industry output is 480 units and industry price will be $p = 1 - .001(480) = 0.52$. Each firm will have profits equal to $\pi = 57.6$ and total industry profits will be 115.2. The equilibrium presented is the only plausible solution to this problem since it is the only one where beliefs

are consistent with results. The difficulty is how the firms arrive at these correct beliefs in a one period, one-shot model.

3.2.6. *Comparison of duopoly with the cartel equilibrium.* If the firms act as a cartel, they will consult and set output such that joint profits are maximized. The problem is set up as follows where only total output is of concern

$$\begin{aligned}
 \pi &= \max_{q_1 + q_2} [p(q_1 + q_2) - C(q_1) - C(q_2)] \\
 &= \max_Q [(A - BQ)Q - cQ] \\
 &= \max_Q [AQ - BQ^2 - cQ] \\
 \Rightarrow \frac{d\pi}{dQ} &= A - 2BQ - c = 0 \\
 \Rightarrow 2BQ &= A - c \\
 \Rightarrow Q &= \frac{A - c}{2B}
 \end{aligned} \tag{29}$$

The equilibrium price is given by

$$\begin{aligned}
 p &= (A - BQ^*) \\
 &= \left(A - B \left(\frac{A - c}{2B} \right) \right) \\
 &= \left(A - \frac{1}{2}A + \frac{1}{2}c \right) \\
 &= \left(\frac{1}{2}A + \frac{1}{2}c \right) \\
 &= \frac{A + c}{2}
 \end{aligned} \tag{30}$$

This is the same as the monopoly solution. If the firms split the output and profit they obtain total profit of

$$\begin{aligned}
 \pi_m &= (A - BQ^*)Q^* - cQ^* \\
 &= \left(A - B \left(\frac{A - c}{2B} \right) \right) Q^* - cQ^* \\
 &= \left(A - \frac{1}{2}A + \frac{1}{2}c \right) Q^* - cQ^* \\
 &= \left(\frac{1}{2}A + \frac{1}{2}c \right) Q^* - cQ^* \\
 &= \left(\frac{1}{2}A - \frac{1}{2}c \right) Q^*
 \end{aligned} \tag{31}$$

If we solve the numerical example directly we obtain

$$\begin{aligned}
\pi &= \max_{q_1 + q_2} [p(q_1 + q_2) - C(q_1) - C(q_2)] \\
&= \max_Q [(1 - .001Q)Q - .28Q] \\
&= \max_Q [Q - .001Q^2 - .28Q]
\end{aligned} \tag{32}$$

$$\begin{aligned}
\Rightarrow \frac{d\pi}{dQ} &= 1 - .002Q - .28 = 0 \\
\Rightarrow .002Q &= .72 \\
\Rightarrow Q &= 360
\end{aligned}$$

At this quantity the market price will be $p = 1 - .001(360) = .64$ which is higher than the duopoly price of .52 and is higher than marginal cost of .28. Thus consumers are better off with the Cournot duopoly as compared to the cartel solution. The total profits in the cartel are given by $\pi = .64(360) - .28(360) = \129.60 . How the firms divide up these profits is rather arbitrary ranging from an equal split to one firm having almost all of them. Because cartel profits are higher than the Cournot profits, there is a definite incentive for the firms to collude.

3.2.7. *Comparison with the competitive equilibrium.* If the firms act competitively they will each take price as given and maximize profits ignoring the other firm. For Firm 1 the supply function is derived as follows

$$\begin{aligned}
\pi_1 &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} [p q_1 - c q_1] \\
\Rightarrow \frac{d\pi}{dq_1} &= p - c = 0 \\
\Rightarrow p &= c
\end{aligned} \tag{33}$$

With constant marginal cost the firm will supply an infinite amount at a price of c and none if price is less. Similarly for the second firm. Thus the equilibrium price will be c . For the numerical example this means that $p = .28$. Market demand and supply will be given by

$$Q = \frac{A - p}{B} = \frac{A - c}{B} \tag{34}$$

For the example market demand is given by $Q = 1000 - 1000p = 720$. Neither firm will make a profit and will be indifferent about production levels. Thus the firms will clearly take account of each other's actions as in the duopoly solution.

3.3. The Cournot model with many firms.

3.3.1. *Inverse demand equations.* The market inverse demand equation is given by

$$\begin{aligned}
p &= A - BQ \\
&= A - B(q_1 + q_2 + \dots + q_N) \\
&= A - B(\sum_{j=1}^N q_j)
\end{aligned} \tag{35}$$

For the i^{th} firm we can write

$$\begin{aligned}
p &= A - BQ \\
&= A - B(q_1 + \cdots + q_{i-1} + q_i + q_{i+1} + \cdots + q_N) \\
&= A - B(\sum_{j \neq i} q_j) - Bq_i \\
&= A - BQ_{-i} - Bq_i, \quad Q_{-i} = \sum_{j \neq i} q_j
\end{aligned} \tag{36}$$

3.3.2. *Profit maximization.* Profit for the i^{th} firm is given by

$$\begin{aligned}
\pi_i &= \max_{q_i} [p q_i - C(q_i)] \\
&= \max_{q_i} [(A - BQ_{-i} - Bq_i) q_i - c q_i] \\
&= \max_{q_i} [A q_i - BQ_{-i} q_i - Bq_i^2 - c q_i]
\end{aligned} \tag{37}$$

Optimizing with respect to q_i will give

$$\begin{aligned}
\frac{d\pi_i}{dq_i} &= A - BQ_{-i} - 2Bq_i - c = 0 \\
\Rightarrow 2Bq_i &= A - BQ_{-i} - c \\
\Rightarrow q_i^* &= \frac{A - c}{2B} - \frac{Q_{-i}}{2}
\end{aligned} \tag{38}$$

Because all firms are identical, this holds for all other firms also.

3.3.3. *Nash equilibrium.* In a Nash equilibrium, each firm i chooses a best response q_i^* that reflects a correct prediction of the other $N-1$ outputs in total. Denote this sum as Q_{-i}^* . A Nash equilibrium is then

$$q_i^* = \frac{A - c}{2B} - \frac{Q_{-i}^*}{2} \tag{39}$$

Because all the firms are identical we can write $Q_{-i}^* = \sum_{i \neq j} q_j^* = (N-1)q^*$ where q^* is the optimal output for the identical firms. We can then write

$$\begin{aligned}
q^* &= \frac{A - c}{2B} - \frac{(N - 1)q^*}{2} \\
\Rightarrow \frac{2q^* + (N - 1)q^*}{2} &= \frac{A - c}{2B} \\
\Rightarrow \frac{(N + 1)q^*}{2} &= \frac{A - c}{2B} \\
\Rightarrow q^* &= \frac{A - c}{(N + 1)B}
\end{aligned} \tag{40}$$

Industry output and price are then given by

$$\begin{aligned}
Q^* &= \frac{N(A-c)}{(N+1)B} \\
p^* &= A - BQ^* \\
&= A - B \left(\frac{N(A-c)}{(N+1)B} \right) \\
&= \frac{(N+1)A}{N+1} - \frac{NA}{(N+1)} - \frac{Nc}{N+1} \\
&= \frac{A}{N+1} + \frac{N}{N+1}c
\end{aligned} \tag{41}$$

Notice that if $N=1$ we get the monopoly solution while if $N=2$ we get the duopoly solution, etc. Consider what happens as $N \rightarrow \infty$. The first term in the expression for price goes to zero and the second term converges to 1. Thus price gets very close to marginal cost. Considering quantities, the ratio $\frac{N(A-c)}{(N+1)B} \rightarrow \frac{(A-c)}{B}$ which is the competitive solution.

3.3.4. *Numerical example.* Let the numerical parameters be given by

$$\begin{aligned}
p &= A - BQ \\
&= 1 - .001Q \\
\Rightarrow A &= 1, B = 0.001 \\
c &= 0.28
\end{aligned} \tag{42}$$

From equation 38 the reaction function for a representative firm in the Cournot example model takes the form

$$\begin{aligned}
q_i^* &= \frac{A-c}{2B} - \frac{1}{2}Q_{-i} \\
&= \frac{1-0.28}{0.002} - \frac{1}{2}q^*(N-1), \quad Q_{-i} = q^*(N-1) \\
&= 360 - \frac{1}{2}q^*(N-1)
\end{aligned} \tag{43}$$

where q^* is the identical output of each of the other firms. The optimal q for each firm will be

$$\begin{aligned}
q^* &= \frac{A-c}{(N+1)B} \\
&= \frac{1-0.28}{(N+1)(0.001)} \\
&= \frac{720}{N+1}
\end{aligned} \tag{44}$$

with equilibrium price given by

$$p^* = \frac{1 + .28N}{N+1} \tag{45}$$

Clearly the larger the number of firms the smaller is output per firm, the higher is industry output, and the lower is price.

3.4. The Cournot model with differing costs.

3.4.1. *Individual firm optima with different marginal costs.* Consider first a two firm model where firm i has marginal cost c_i . The optimal solution is obtained as before by defining profit and choosing q_i to maximize profit given the level of output of the other firm. Consider the first firm as follows.

$$\begin{aligned}
 \pi_1 &= \max_{q_1} [p q_1 - C(q_1)] \\
 &= \max_{q_1} [(A - B q_1 - B q_2) q_1 - c_1 q_1] \\
 &= \max_{q_1} [A q_1 - B q_1^2 - B q_1 q_2 - c_1 q_1] \\
 \Rightarrow \frac{d\pi_1}{dq_1} &= A - 2B q_1 - B q_2 - c_1 = 0 \\
 \Rightarrow 2B q_1 &= A - B q_2 - c_1 \\
 \Rightarrow q_1^* &= \frac{A - c_1}{2B} - \frac{q_2}{2}
 \end{aligned} \tag{46}$$

Similarly the best response function for the second firm is given by

$$q_2^* = \frac{A - c_2}{2B} - \frac{q_1}{2} \tag{47}$$

3.4.2. *Nash equilibrium.* We solve for the optimal q_i^* simultaneously as in the equivalent marginal cost case.

$$\begin{aligned}
 q_2^* &= \frac{A - c_2}{2B} - \frac{q_1}{2} \\
 &= \frac{A - c_2}{2B} - (0.5) \left(\frac{A - c_1}{2B} - \frac{q_2}{2} \right) \\
 &= \frac{0.5A - c_2 + 0.5c_1}{2B} + \frac{0.5q_2}{2} \\
 &= \frac{A - 2c_2 + c_1}{4B} + \frac{q_2}{4} \\
 \Rightarrow \frac{3q_2^*}{4} &= \frac{A - 2c_2 + c_1}{4B} \\
 \Rightarrow q_2^* &= \frac{A - 2c_2 + c_1}{3B}
 \end{aligned} \tag{48}$$

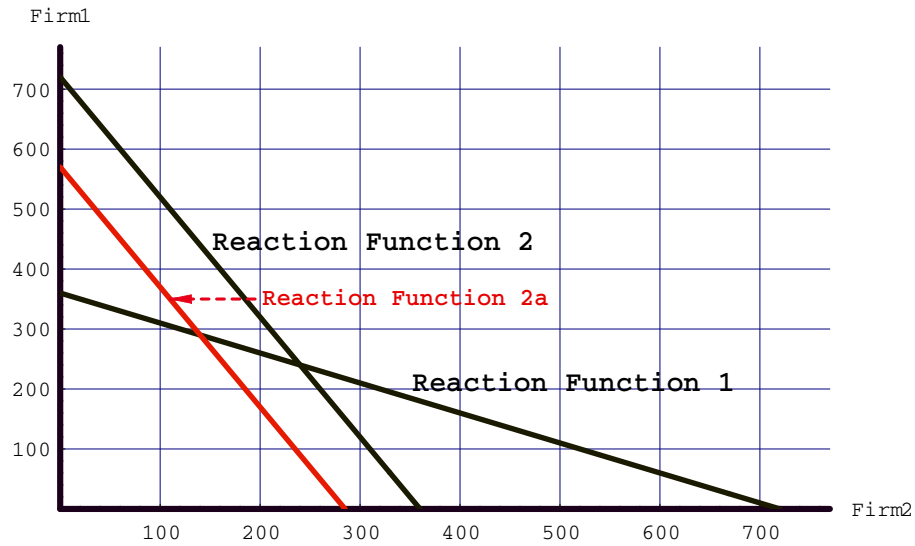
We can solve for q_1^* in a similar fashion to obtain

$$q_1^* = \frac{A - 2c_1 + c_2}{3B} \tag{49}$$

Graphically consider a case where $c_1 = c$ and $c_2 > c$ in figure 4. Notice that the response function of Firm 2 has moved inward so that it produces less for each value of Firm 1. Its optimal output is less.

For the numerical example and a marginal cost for the second firm of \$0.43 with the first firm's marginal cost remaining at \$0.28, we obtain

FIGURE 4. Cournot Equilibrium from Two Firms with Different Marginal Costs



$$\begin{aligned}
 q_1^* &= \frac{A - 2c_1 + c_2}{3B} \\
 &= \frac{1 - 2(.28) + .43}{3(.001)} \\
 &= 290 \\
 q_2^* &= \frac{A - 2c_2 + c_1}{3B} \\
 &= \frac{1 - 2(.43) + .28}{3(.001)} \\
 &= 140 \\
 p &= .57
 \end{aligned} \tag{50}$$

At this price each firm has marginal revenue equal to marginal cost. Total production of 430 is less than the 480 that occurred when both firms had marginal cost of \$0.28.

3.4.3. *Cournot equilibrium in model with many firms and non-identical costs.* Consider now the case where each of N firms has its own marginal cost function. We can find the optimum for the i^{th} firm as follows.

$$\begin{aligned}
\pi_i &= \max_{q_i} [p q_i - C(q_i)] \\
&= \max_{q_i} [(A - B Q_{-i} - B q_i) q_i - c_i q_i] \\
&= \max_{q_i} [A q_i - B Q_{-i} q_i - B q_i^2 - c_i q_i] \\
\Rightarrow \frac{d\pi_i}{dq_i} &= A - B Q_{-i} - 2 B q_i - c_i = 0
\end{aligned} \tag{51}$$

The values that satisfy the first order condition are the equilibrium values. Denote them by Q_{-i}^* and q_i^* . The sum of these two components is $Q^* = Q_{-i}^* + q_i^*$. Now rearrange the first order condition to obtain

$$\begin{aligned}
A - B Q_{-i}^* - 2 B q_i^* - c_i &= 0 \\
\Rightarrow A - B Q_{-i}^* - B q_i^* - c_i &= B q_i^* \\
\Rightarrow A - B Q^* - c_i &= B q_i^*
\end{aligned} \tag{52}$$

Note that $p^* = A - B Q^*$ and then rewrite equation 52 as follows

$$\begin{aligned}
A - B Q^* - c_i &= B q_i^* \\
\Rightarrow p^* - c_i &= B q_i^*
\end{aligned} \tag{53}$$

Now multiply both sides of equation 53 by $\frac{1}{p^*}$ and multiply the right hand side by $\frac{Q^*}{Q^*}$ to obtain

$$\begin{aligned}
p^* - c_i &= B q_i^* \\
\frac{p^* - c_i}{p^*} &= \frac{B q_i^* Q^*}{p^* Q^*} \\
\Rightarrow \frac{p^* - c_i}{p^*} &= \frac{B Q^*}{p^*} \frac{q_i^*}{Q^*} \\
&= \frac{B Q^*}{p^*} s_i
\end{aligned} \tag{54}$$

where s_i is the share of the i th firm in total equilibrium output. The left hand side of this expression is the Lerner index of monopoly power. With a linear demand curve, the first term in the right hand side is the reciprocal of the elasticity of demand. To see this write the demand equation as

$$\begin{aligned}
p &= A - BQ \\
Q &= \frac{A}{B} - \frac{p}{B} \\
\frac{dQ}{dp} &= \frac{-1}{B} \\
\epsilon &= \frac{dQ}{dp} \frac{p}{Q} \\
&= \frac{-1}{B} \frac{p}{Q}
\end{aligned} \tag{55}$$

Hence we can write

$$\frac{p^* - c_i}{p^*} = \frac{-s_i}{\epsilon} \quad (56)$$

Thus firms with a larger share in an industry with a more inelastic demand will have more market power.

4. THE BERTRAND MODEL

4.1. Introduction to the Bertrand Model. In the Cournot model each firm independently chooses its output. The price then adjusts so that the market clears and the total output produced is bought. Yet, upon reflection, this phrase “the price adjusts so that the market clears” appears either vague or incomplete. What exactly does it mean? In the context of perfectly competitive markets, the issue of price adjustment is perhaps less pressing. A perfectly competitive firm is so small that its output has no effect on the industry price. From the standpoint of the individual competitive firm, prices are given i.e., it is a “price-taker.” Hence, for analyzing competitive firm behavior, the price adjustment issue does not arise. The issue of price adjustment *does* arise, however, from the perspective of an entire competitive industry. That is, we are obliged to say something about where the price, which each individual firm takes as given, comes from. We usually make some assumption about the “invisible hand” or the “Walrasian auctioneer”. This mechanism is assumed to work impersonally to insure that the price is set at its market clearing level. But in the Cournot model, especially when the number of firms is small, reliance upon the fictional auctioneer of competitive markets seems strained. After all, the firms clearly recognize their interdependence. Far from being a price-taker, each firm is keenly aware that the decisions it makes will affect the industry price. In such a setting, calling upon the auctioneer to set the price is a bit inconsistent with the development of the underlying model. Indeed, in many circumstances, it is more natural “to cut to the chase” directly and assume that firms compete by setting prices and *not* quantities. Consumers then decide how much to buy at those prices.

The Cournot duopoly model, recast in terms of price strategies rather than quantity strategies, is referred to as the Bertrand model. Joseph Bertrand was a French mathematician who reviewed and critiqued Cournot’s work nearly fifty years after its publication in 1883, in an article in the *Journal des Savants*. A central point in Bertrand’s review was that the change from quantity to price competition in the Cournot duopoly model led to dramatically different results.

4.2. The Basic Bertrand Model.

4.2.1. Assumptions of the basic Bertrand Model.

1. Two firms with no additional entry.
2. Homogeneous product such that $q_1 + q_2 = Q$ where Q is industry output and q_i is the output of the i th firm.
3. Single period of production and sales
4. Market and inverse market demand is a linear function of price.

$$\begin{aligned} p &= A - BQ = A - B(q_1 + q_2) \\ Q &= \frac{A}{B} - \frac{p}{B} = \frac{A - p}{B} \end{aligned} \quad (57)$$

5. Each firm has constant and equal marginal cost equal to c . With constant marginal cost, average cost is also constant and equal to c .
6. The decision variable is the price to charge for the product.

4.2.2. *Residual demand curves in the Bertrand model.* If Firm 2 believes that Firm 1 will sell q_1 units of output, then its residual inverse demand curve is given by

$$p = A - Bq_1 - Bq_2 = (A - Bq_1) - Bq_2 \quad (58)$$

while residual demand is given by

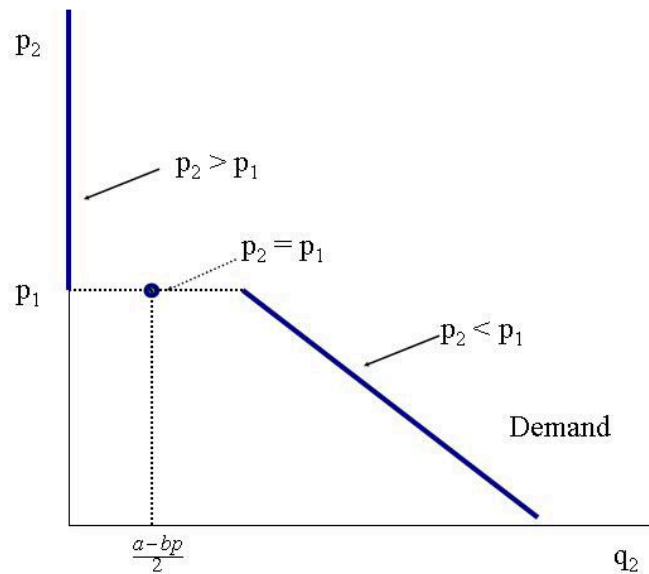
$$Q = \frac{A}{B} - \frac{p}{B} = a - bp \text{ where } a = \frac{A}{B} \text{ and } b = \frac{1}{B} \quad (59)$$

In order to determine its best price choice, Firm 2 must first work out the demand for its product *conditional* on both its own price, p_2 , and Firm 1's price, p_1 . Firm 2's reasoning then proceeds as follows. If $p_2 > p_1$, Firm 2 will sell no output. The product is homogenous and consumers always buy from the cheapest source. Setting a price above that of Firm 1, means Firm 2 serves no customers. The opposite is true if $p_2 < p_1$. When Firm 2 sets the lower price, it will supply the entire market, and Firm 1 will sell nothing. Finally, we assume that if $p_2 = p_1$, the two firms split the market. The foregoing implies that the demand for Firm 2's output, q_2 , may be described as follows:

$$\begin{aligned} q_2 &= 0 \text{ if } p_2 > p_1 \\ q_2 &= a - bp_2 \text{ if } p_2 < p_1 \\ q_2 &= \frac{(a - bp_1)}{2} \text{ if } p_2 = p_1 \end{aligned} \quad (60)$$

Consider a graphical representation of this demand function in figure 5

FIGURE 5. Demand for Firm 2 in Bertrand Model



The demand structure is *not* continuous. For any p_2 greater than p_1 , demand for q_2 is zero. But when p_2 falls and becomes equal to p_1 , demand jumps from zero to $(a - bp_1)/2$. When p_2 then falls still further

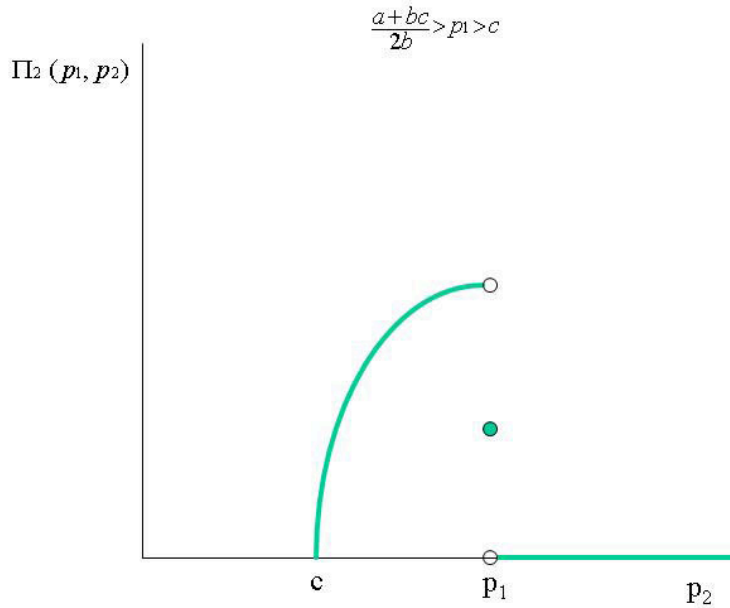
so that it is below p_1 , demand then jumps again to $a - bp_2$. This discontinuity in Firm 2's demand curve is not present in the quantity version of the Cournot model. This discontinuity in demand carries over into a discontinuity in profits. Firm 2's profit, Π_2 , as a function of p_1 and p_2 is given by:

$$\Pi_2(p_1, p_2) = \begin{cases} 0, & p_2 > p_1 \\ (p_2 - c)(a - bp_2), & p_2 < p_1 \\ \frac{(p_2 - c)(a - bp_2)}{2}, & p_2 = p_1 \end{cases} \quad (61)$$

4.2.3. *Best response functions in the Bertrand model.* To find Firm 2's best response function, we need to find the price, p_2 , that maximizes Firm 2's profit, $\Pi_2(p_1, p_2)$, for any given choice of p_1 . For example, suppose that Firm 1 chose a very high price— higher even than the pure monopoly price which is $p^M = \frac{a+bc}{2b} = \frac{\frac{A}{b} + \frac{c}{b}}{2} = \frac{A+c}{2}$. Because Firm 2 can capture the entire market by selecting any price lower than p_1 , its best bet would be to choose the pure monopoly price, p^M , and thereby earn the pure monopoly profits.

Conversely, what if Firm 1 set a very low price, say one below its unit cost, c ? If $p_1 < c$, Firm 2 is best setting its price at some level above p_1 . This will mean, we know, that Firm 2 will sell nothing and earn zero profits. What about the more likely case in which Firm 1 sets its price above marginal cost, c , but below the pure monopoly price, p^M ? How should Firm 2 optimally respond in these circumstances? The simple answer is that it should set a price *just a bit less than* p_1 . The intuition behind this strategy is illustrated in figure 6 which shows Firm 2's profits given a price, p_1 , satisfying $(a+bc)/2b > p_1 > c$.

FIGURE 6. Profit for Firm 2 in Bertrand Model



Observe that Firm 2's profits rise continuously as p_2 rises from c to just below p_1 . Whenever p_2 is less than p_1 , Firm 2 is the only company that any consumer buys from. However, in the case where p_1 is less

than p^M , the monopoly position that Firm 2 obtains from undercutting p_1 is constrained. In particular, it cannot achieve the pure monopoly price, p^M , and associated profits, because at that price, Firm 2 would lose all its customers. Still, the firm will wish to get as close to that result as possible. It could, of course, just match Firm 1's price exactly. But whenever it does so, it shares the market equally with its rival. If, instead of setting $p_2 = p_1$, Firm 2 just *slightly* reduces its price below the p_1 level, it will double its sales while incurring only a infinitesimal decline in its profit margin per unit sold. This is a trade well worth the making as the figure makes clear. In turn, the implication is that for any p_1 such that $c < p_1 < p^M$, Firm 2's best response is to set $p_2 = p_1 - \varepsilon$, where ε is an arbitrarily small amount.

The last case to consider is the case in which Firm 1 prices at cost so that $p_1 = c$. Clearly, Firm 2 has no incentive to undercut this value of p_1 . To do so, would only involve Firm 2 in losses. Instead, Firm 2 will do best to set p_2 either equal to or above p_1 . If it prices above p_1 , Firm 2 will sell nothing and earn zero profits. If it matches p_1 , it will enjoy positive sales but break even on every unit sold. Accordingly, Firm 2 will earn zero profits in this latter case, too. Thus, when $p_1 = c$, Firm 2's best response is to set p_2 either greater than or equal to p_1 .

Now let p_2^* denote Firm 2's best response price for any given value of p_1 . Our preceding discussion may be summarized as follows:

$$p_2^* = \begin{cases} \frac{a+bc}{2b}, & p_1 > \frac{a+bc}{2b} \\ p_1 - \varepsilon, & c < p_1 \leq \frac{a+bc}{2b} \\ \geq p_1, & c = p_1 \\ > p_1, & c > p_1 \geq 0 \end{cases} \quad (62)$$

By similar reasoning, Firm 1's best response, p_1^* , for any given value of p_2 is given by:

$$p_1^* = \begin{cases} \frac{a+bc}{2b}, & p_2 > \frac{a+bc}{2b} \\ p_2 - \varepsilon, & c < p_2 \leq \frac{a+bc}{2b} \\ \geq p_2, & c = p_2 \\ > p_2, & c > p_2 \geq 0 \end{cases} \quad (63)$$

4.2.4. Equilibrium in the Bertrand model. We are now in a position to determine the Nash equilibrium for the duopoly when played in prices. We know that a Nash equilibrium is one in which each firm's expectation regarding the action of its rival is precisely the rival's best response to the strategy chosen by the firm in question in anticipation of that response. For example, the strategy combination, $[p_1 = \frac{a+bc}{2b}, p_2 = \frac{a+bc}{2b} - \varepsilon]$ cannot be an equilibrium. This is because in that combination, Firm 2 is choosing to undercut Firm 1 on the expectation that Firm 1 chooses the monopoly price. But Firm 1 would only choose the monopoly price if it thought that Firm 2 was going to price *above* that level. In other words, for the suggested candidate equilibrium, Firm 1's strategy is *not* a best response to Firm 2's choice. Hence, this strategy combination cannot be a Nash equilibrium.

There is, essentially, only one Nash equilibrium for the Bertrand duopoly game where both firms are producing. It is the price pair, $(p_1^* = c, p_2^* = c)$. If Firm 1 sets this price in the expectation that Firm 2 will do so, and if Firm 2 acts in precisely the same manner, neither will be disappointed. Hence, the outcome of the Bertrand duopoly game is that the market price equals marginal cost. This is, of course, exactly what occurs under perfect competition. The only difference is that here, instead of many small firms, we have just two, large ones.

4.2.5. Criticisms of the Bertrand model.

1. Small changes in price lead to dramatic changes in quantity

The chief criticism of the Bertrand model is its assumption that *any* price deviation between the two firms leads to an immediate and total loss of demand for the firm charging the higher price. It is this assumption that gives rise to the discontinuity in either firm's demand and profit functions. It is also this assumption that underlies the derivation of each firm's best response function.

There are two reasons why one firm's decision to charge a price somewhat higher than its rival may not cause it to lose all its customers. One of these factors is the existence of capacity constraints.¹ The other is that the two goods may not be identical.

2. Capacity constraints

Consider a small town with two feed mills. Suppose that both mills are currently charging a fee of \$4.85 per cwt for mixed feed. According to the Bertrand model, one mill, say Firm 2 should believe that any fee that it sets below \$4.85 for the same service will enable it to steal instantly all of Firm 1's customers. But suppose, for example, both firms were initially doing a business of 20 customers per day and each had a maximum capacity of 30 customers per day, Firm 2 would not be able to service all the demand implied by the Bertrand analysis at a price of \$4.75.

More generally, denote as Q^C , the competitive output or the total demand when price is equal to marginal cost i.e., $Q^C = a - bc$. If neither firm has the capacity to produce Q^C (neither could individually meet the total market demand generated by competitive prices), but instead, each can produce only a smaller amount, then the Bertrand outcome with $p_1 = p_2 = c$ will *not* be the Nash equilibrium. In the Nash equilibrium, it must be the case that each firm's choice is a best response to strategy of the other. Consider then the original Bertrand solution with prices chosen to be equal to marginal cost, c , and profit at each firm equal to zero. Because there is now a capacity constraint, either firm, say Firm 2 for instance, can contemplate *raising* its price. If Firm 2 sets p_2 above marginal cost, and hence, above p_1 , it would surely lose some customers. But it would not lose all of its customers. Firm 1 *does not have the capacity* to serve them. Some customers would remain with Firm 2. Yet Firm 2 is now earning some profit from each such customer ($p_2 > c$). So, its total profits are now positive whereas before they were zero. It is evident therefore that $p_2 = c$, is not a best response to $p_1 = c$. So, the strategy combination ($p_1 = c, p_2 = c$) cannot be a Nash equilibrium, if there are binding capacity constraints.

3. Homogeneity of products

The second reason that the Bertrand solution may fail to hold is that the two firms do not, as assumed, produce identical products. The two feed mills in the example may not produce exactly the same quality of feed. Indeed, as long as the two firms are not side-by-side, they differ in their location and customers may prefer one or the other based on distance from their own operation.

¹Edgeworth (1987) was one of the first economists to investigate the impact of capacity constraints on the Bertrand analysis.

5. THE BERTRAND MODEL WITH SPATIAL PRODUCT DIFFERENTIATION

5.1. **The idea of a spatial model.** We can often capture the nature of product differentiation using a spatial model. This idea is originally due to Hotelling. The idea is to set up a market in one dimension where the customers are uniformly distributed in this dimension. The easiest case is to consider a town with a main street of unit (say one mile) length. This market is supplied by two firms with different owners and management. One firm—located at the west end of town—has the address $x = 0$. The other—located at the east end of town has the location, $x = 1$. Each firm has the same, constant unit cost of production c . Each point on the line is associated with an x value measuring the location of that point relative to the west or left end of town. A consumer whose most preferred product or location is x^j is called consumer x^j . While consumers differ about which variant of the good is best, they are the same in that each has the same reservation demand price, V . Each also will buy at most one unit of the product. If a farmer, for example, purchases feed at a location located “far away” from her most preferred location, she incurs a utility cost. In particular, firm x^j incurs the cost tx^j if it purchases good 1 (located at $x=0$), and the cost $t(1-x^j)$ if it purchases good 2 (located at $x=1$). Figures 7 and 8 describe the market setting.

FIGURE 7. Main Street Spatial Model

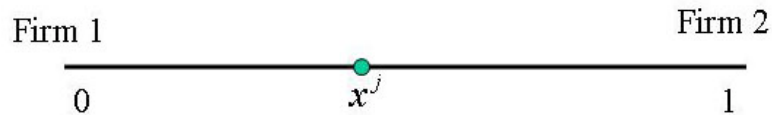
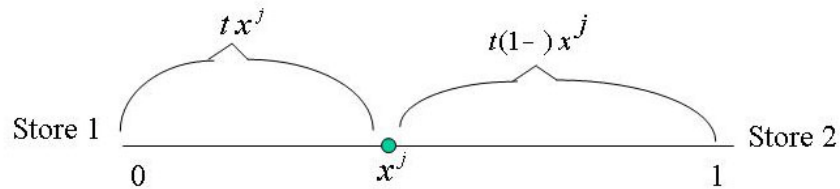
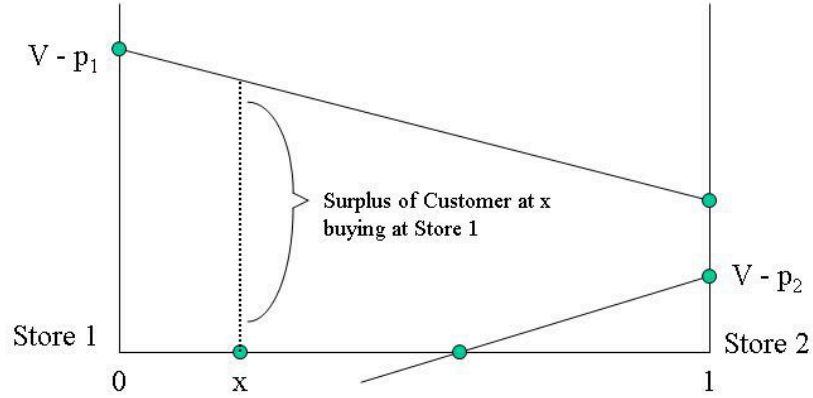


FIGURE 8. Main Street Spatial Model with Transportation Costs



5.2. **The Bertrand model as a pricing game.** Assume the selling firms compete in prices, p_1 and p_2 , respectively. The prices are chosen simultaneously. as the solution to the game. One requirement for a Nash equilibrium is that both firms have a positive market share. If this condition were not satisfied it would mean that at least one firm’s price was so high that it had zero market share and, therefore, zero profits. Since such a firm could always obtain positive profits by cutting its price, such a situation cannot be part of a Nash equilibrium. To simplify assume that the Nash equilibrium outcome is one in which the entire market is served. That is, assume the outcome involves a market configuration in which every consumer buys the product from either Firm 1 or Firm 2. This assumption will be true so long as the reservation price,

FIGURE 9. Surplus for Each Customer In Spatial Model with $p_2 \gg p_1$ 

V , is sufficiently large. The surplus enjoyed by a consumer at point x when buying from Firm 1 is shown in figure 9

When V is large, firms will have an incentive to sell to as many customers as possible because such a high willingness-to-pay will imply that each customer can be charged a price sufficiently high to make each such sale profitable. If V is not high relative to the prices charges, the market may not be served as in figure 10

An important implication of the assumption that the entire market is served is that the marginal consumer x^m , is indifferent between buying from either Firm 1 or Firm 2. That is, she enjoys the same surplus either way. Algebraically, for the marginal consumer we have we have:

$$V - p_1 - tx^m = V - p_2 - t(1 - x^m) \quad (64)$$

Equation 64 may be solved to find the address of the marginal consumer, x^m . This is

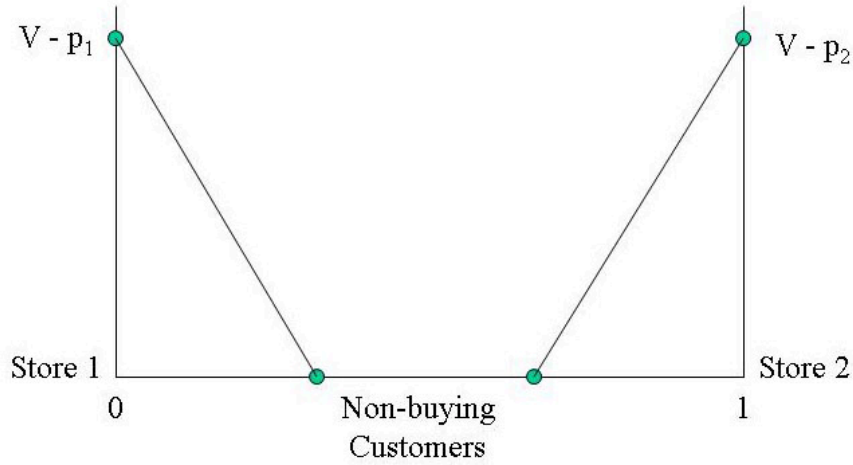
$$x^m(p_1, p_2) = \frac{p_2 - p_1 + t}{2t} \quad (65)$$

At any set of prices, p_1 and p_2 , all consumers to the west or left of x^m buy from Firm 1. All those to the east or right of x^m buy from Firm 2. In other words, x^m is the fraction of the market buying from Firm 1 and $(1 - x^m)$ is the fraction buying from Firm 2. If the total market size is N , the demand function facing Firm 1 at any price combination (p_1, p_2) in which the entire market is served is:

$$D^1(p_1, p_2) = x^m(p_1, p_2) N = \left[\frac{(p_2 - p_1 + t)}{2t} \right] N \quad (66)$$

Similarly, Firm 2's demand function is:

FIGURE 10. Surplus to Customers In Spatial Model with $p_2 = p_1$ and both Prices High Relative to V



$$D^2(p_1, p_2) = [1 - x^m(p_1, p_2)] N = \left[\frac{(p_1 - p_2 + t)}{2t} \right] N \quad (67)$$

Unlike the original Bertrand duopoly model, the model here is one in which the demand function facing either firm *is* continuous in both p_1 and p_2 . This is because when goods are differentiated, a decision by say Firm 1 to set p_1 a little higher than its rival's price, p_2 , does not cause Firm 1 to lose all of its customers. Some of its customers will still prefer to buy good 1 even at the higher price simply because they prefer that version of the good to the type (or location) marketed by Firm 2.²

The continuity in demand functions carries over into the profit functions. Firm 1's profit function is:

$$\Pi^1(p_1, p_2) = N(p_1 - c) \left(\frac{p_2 - p_1 + t}{2t} \right) \quad (68)$$

Similarly, Firm 2's profits are given by:

$$\Pi^2(p_1, p_2) = N(p_2 - c) \left(\frac{p_1 - p_2 + t}{2t} \right) \quad (69)$$

We may derive Firm 1's best response function by differentiating Π^1 with respect to p_1 taking p_2 as given. In this way, we will obtain Firm 1's best price choice, p_1^* as a function of p_2 . Of course, differentiation for Firm 2's profit function with respect to its price will, similarly, yield the best choice of p_2 conditional upon any choice of p_1 . The firms are symmetrical so their best response functions are mirror images of each other. Specifically, the best response function for firm one is

²The assumption that the equilibrium is one in which the entire market is served is critical to the continuity result.

$$\begin{aligned}
\Pi^1(p_1, p_2) &= N(p_1 - c) \left(\frac{p_2 - p_1 + t}{2t} \right) \\
\frac{\partial \Pi^1(p_1, p_2)}{dp_1} &= N(p_1 - c) \left(\frac{-1}{2t} \right) + \left(\frac{p_2 - p_1 + t}{2t} \right) N = 0 \\
\Rightarrow \left(\frac{-N p_1}{2t} \right) - \left(\frac{-N p_1}{2t} \right) &= N \left(\frac{-c - p_2 - t}{2t} \right) \\
\Rightarrow \frac{-2 p_1}{2t} &= \left(\frac{-c - p_2 - t}{2t} \right) \\
\Rightarrow p_1^* &= \left(\frac{p_2 + c + t}{2} \right)
\end{aligned} \tag{70}$$

Similarly, the best response function for firm two is

$$p_2^* = \left(\frac{p_1 + c + t}{2} \right) \tag{71}$$

The Nash equilibrium is a pair of best response prices, p_1^* , p_2^* , such that p_1^* is Firm 1's best response to p_2^* , and p_2^* is Firm 2's best response to p_1^* . This means that we may replace p_1 and p_2 on the right-hand-side of equations 70 and 71 with p_1^* and p_2^* , respectively. Then, solving jointly for the Nash equilibrium pair, (p_1^*, p_2^*) yields

$$\begin{aligned}
p_2^* &= \left(\frac{p_1 + c + t}{2} \right) \\
&= \left(\frac{\left(\frac{p_2 + c + t}{2} \right) + c + t}{2} \right) \\
&= \frac{p_2}{4} + \frac{c}{4} + \frac{t}{4} + \frac{c}{2} + \frac{t}{2} \\
\Rightarrow \frac{3}{4} p_2^* &= \frac{3}{4} c + \frac{3}{4} t \\
\Rightarrow p_2^* &= c + t
\end{aligned} \tag{72}$$

Substituting for p_2^* we obtain

$$\begin{aligned}
p_1^* &= \left(\frac{p_2 + c + t}{2} \right) \\
&= \left(\frac{c + t + c + t}{2} \right) \\
&= \frac{2(c + t)}{2} \\
\Rightarrow p_1^* &= c + t
\end{aligned} \tag{73}$$

In other words, the symmetric Nash equilibrium for the Bertrand duopoly model with differentiated products is one in which each firm charges the same price given by its unit cost *plus* an amount, t , the utility

cost per unit of distance a consumer incurs in buying a good different from her own most preferred type. At these prices, the firms split the market. The marginal consumer is at $x^m = \frac{1}{2}$. Profits at each firm are $\frac{Nt}{2}$.

Two points are worth making in connection with the foregoing analysis. First, note the role that the parameter, t , plays. t is a measure of the value each consumer places on obtaining her most preferred version of the product. The greater is t , the more the consumer is willing to pay a high price simply to avoid being “far away” from her favorite style. That is, a high t value indicates that either firm has little to worry about in charging a high price because the consumers would prefer to pay that price rather than buy a low-price alternative that is “far away” from their preferred style. Thus, when t is large, the price competition between the two firms is softened. A large value of t implies then that the product differentiation matters and that, to the extent products are differentiated, price competition will be less intense. However, as t falls the consumers place less value on obtaining a preferred type of product and focus more on simply obtaining the best price. This intensifies price competition. In the limit, when $t=0$, differentiation is of no value to the consumers. They treat all goods as essentially identical. Price competition becomes fierce and, in the limit, forces prices to be set at marginal cost just as in the original Bertrand model.

The second point to be made in connection with the above analysis concerns the location of the firms. We simply assumed that the two firms were located at either end of town. It turns out allowing the firms in the above model to choose *both* their price and their location strategies makes the problem quite complicated. Still, the intuition behind this indeterminacy is instructive. Two opposing forces make the combined choice of price and location difficult. On the one hand, the two firms will wish to avoid locating at the same point because to do so, eliminates any gains from products. Price competition in this case will be fierce as in the original Bertrand model. On the other hand, each firm has an incentive to locate near the center of town. This enables a firm to reach as large a market as possible. Evaluating the balance of these two forces is what makes determination of the ultimate equilibrium so difficult.³

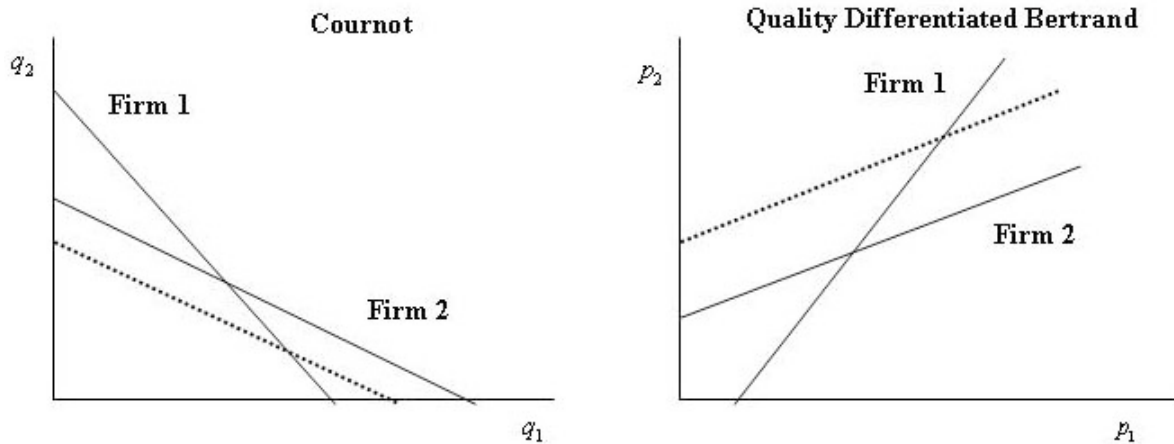
6. STRATEGIC COMPLEMENTS AND SUBSTITUTES IN COURNOT AND BERTRAND MODELS

Best response functions in simultaneous-move games are useful devices for understanding what we mean by a Nash equilibrium outcome. But an analysis of such functions also serves other useful purposes. In particular, examining the properties of best response functions can aid our understanding of how strategic interaction works and how that interaction can be made “more” or “less” competitive.

6.1. Best Response Functions for Cournot and Bertrand Models. Figure 11 shows both the best response curves for the original Cournot duopoly model and the best response functions for the Bertrand duopoly model with differentiated products. One feature in the diagram is immediately apparent. The best response functions for the Cournot quantity model are negatively sloped—Firm 1’s best response to an increase in q_2 is to *decrease* q_1 . But the best response functions in the Bertrand price model are positively sloped. Firm 1’s best response to an increase in p_2 is to increase p_1 , as well. Whether the reaction functions are negatively or positively sloped is quite important. It reveals much about the nature of competition in the two models. To see this, consider the impact of an increase in Firm 2’s unit cost c_2 . Our analysis of the Cournot model indicated that the effect of a rise in c_2 would be to shift *inward* Firm 2’s best response curve. As the figure indicates, this leads to a new Nash equilibrium in which Firm 2 produces less and Firm 1 produces more than each did before c_2 rose. That is, in the Cournot quantity model, Firm 1’s response to Firm 2’s bad luck is a rather aggressive one in which it seizes the opportunity to expand its market share at the expense of Firm 2.

³There is a wealth of literature on this topic with the outcome often depending on the precise functional forms assumed. See, for example, Eaton (1976), D’Aspremont, Gabszewicz, and Thisse (1979), Novshek (1980), and Economides (1989).

FIGURE 11. Best Response Functions for Cournot and Quality Differentiated Bertrand Models



Consider now the impact of a rise in c_2 in the context of the differentiated good Bertrand model. The rise shifts *out* Firm 2's best response function. Given the rise in its costs, Firm 2 now finds it better to set a higher p_2 than it did previously for any given value of p_1 . How does Firm 1 respond? Unlike the Cournot case, Firm 1's reaction is far from one of aggressive opportunism. Quite the contrary, Firm 1—seeing that Firm 2 is now less able to set a low price—realizes that the price competition from Firm 2 is now less intense. Hence, Firm 1 reacts in this case by raising p_1 .

6.2. Strategic complements and substitutes. When the best response functions are upward sloping, we say that the strategies (prices in the Bertrand case) are *strategic complements*. When we have the alternative case of downward sloping response functions, we say that the strategies (quantities in the Cournot case) are *strategic substitutes*. This terminology reflects similar terminology in consumer demand theory. When a consumer reacts to a change in the price of one good by buying either more or less of *both* that good and another product, we say that the two goods are complements. But when a consumer reacts to a change in the price of one product by buying more (less) of it and less (more) of another, we say that the two goods are substitutes. This is the source of the similarity. Prices above are called strategic complements because a change (the rise in c_2) inducing an increase in p_2 also induces an increase in p_1 . Similarly, quantities in Cournot analysis are strategic substitutes because here, such a change in c_2 induces a fall in q_2 but a rise in q_1 .

7. THE STACKELBERG MODEL

7.1. Assumptions of The Stackelberg Model.

1. Two firms with no additional entry.
2. Homogeneous product such that $q_1 + q_2 = Q$ where Q is industry output and q_i is the output of the i^{th} firm.
3. Single period of production and sales
4. Market and inverse market demand is a linear function of price.

$$\begin{aligned}
 p &= A - BQ = A - B(q_1 + q_2) \\
 Q &= \frac{A}{B} - \frac{p}{B} = \frac{A - p}{B}
 \end{aligned}
 \tag{74}$$

5. Each firm has constant and equal marginal cost equal to c . With constant marginal cost, average cost is also constant and equal to c .
6. The decision variable in the quantity of output to produce. One firm acts first and the other firm follows. The leader (Firm 1 for the analysis) assumes that the follower will follow his best response function. Therefore the leader will set output taking this into account. In particular, the leader will use the follower's response function to determine the residual demand curve.

7.1.1. *Example Model.* Assume market demand is given by

$$Q(p) = 1000 - 1000p \tag{75}$$

This implies that inverse demand is given by

$$\begin{aligned}
 Q(p) &= 1000 - 1000p \\
 \Rightarrow 1000p &= 1000 - Q \\
 \Rightarrow p &= 1 - 0.001Q \\
 &= 1 - .001(q_1 + q_2) \\
 &= 1 - 0.001q_1 - 0.001q_2.
 \end{aligned}
 \tag{76}$$

With this demand function, when $p = \$1.00$, $Q = 0$ and when $p = 0$, $Q = 1000$. Also notice that for $p = A - BQ$ to have a positive price with $A > 0$ and $B > 0$ it must be that $\frac{A}{B} > Q$.

For this example, assume that

$$AC = MC = \$0.28. \tag{77}$$

7.2. Residual demand curves in the Stackelberg Model. The residual inverse demand curve for Firm 2 is given by

$$p = A - Bq_1 - Bq_2 = (A - Bq_1) - Bq_2 \tag{78}$$

while the residual inverse demand curve for Firm 1 is given by

$$p = A - Bq_1 - Bq_2 = (A - Bq_2) - Bq_1 \tag{79}$$

7.3. Profit maximization in Stackelberg Model. Consider the profit maximization problem for Firm 2. Firm 2 will maximize profits taking the output of Firm 1 as given. Because Firm 1 moves first, Firm 2 can take this quantity as given. This makes the problem a sequential as opposed to a simultaneous move game. The profit maximization problem for Firm 2 gives

$$\begin{aligned}
\pi_2 &= \max_{q_2} [p q_2 - C(q_2)] \\
&= \max_{q_2} [(A - B q_1 - B q_2) q_2 - c q_2] \\
&= \max_{q_2} [A q_2 - B q_1 q_2 - B q_2^2 - c q_2]
\end{aligned} \tag{80}$$

$$\begin{aligned}
\Rightarrow \frac{d\pi_2}{dq_2} &= A - B q_1 - 2 B q_2 - c = 0 \\
\Rightarrow 2 B q_2 &= A - B q_1 - c \\
\Rightarrow q_2^* &= \frac{A - c}{2 B} - \frac{q_1}{2}
\end{aligned}$$

This is the reaction function for Firm 2. For the numerical example we obtain

$$\begin{aligned}
q_2^* &= \frac{A - c}{2 B} - \frac{q_1}{2} \\
&= \frac{1 - .28}{(2)(.001)} - 0.5 q_1 \\
&= \frac{.72}{.002} - 0.5 q_1 \\
&= 360 - 0.5 q_1
\end{aligned} \tag{81}$$

Now consider the profit maximization problem for Firm 1. Firm 1 realizes that once it chooses its output, q_1 , the other firm (Firm 2) will use its best response function to pick its optimal output $q_2 = R_2(q_1)$. Consider the residual inverse demand for the leader firm (Firm 1) which is given by

$$p = A - B q_1 - B q_2 \tag{82}$$

Now substitute in the optimal response function for Firm 2 to obtain

$$\begin{aligned}
p &= A - B q_1 - B \left(\frac{A - c}{2 B} - \frac{q_1}{2} \right) \\
&= A - B q_1 - \frac{A + c}{2} + \frac{B q_1}{2} \\
&= \frac{A + c}{2} - B \left(\frac{q_1}{2} \right)
\end{aligned} \tag{83}$$

The numerical example simplifies as

$$\begin{aligned}
p &= A - B q_1 - B q_2 \\
&= 1 - .001 q_1 - .001 q_2 \\
&= 1 - .001 q_1 - .001 (360 - 0.5 q_1) \\
&= 1 - .36 - .001 q_1 + .0005 q_1 \\
&= 0.64 - .0005 q_1
\end{aligned} \tag{84}$$

Now write the profit maximization problem for Firm 1 as

$$\begin{aligned}
\pi_1 &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} \left[\left(\frac{A+c}{2} - B \left(\frac{q_1}{2} \right) \right) q_1 - c q_1 \right] \\
&= \max_{q_1} \left[\left(\frac{A+c}{2} \right) q_1 - B \left(\frac{q_1^2}{2} \right) - c q_1 \right] \\
\Rightarrow \frac{d\pi_1}{dq_1} &= \left(\frac{A+c}{2} \right) - B q_1 - c = 0 \\
\Rightarrow B q_1 &= \left(\frac{A+c}{2} \right) - c \\
\Rightarrow B q_1 &= \left(\frac{A+c-2c}{2} \right) \\
\Rightarrow q_1^* &= \frac{A-c}{2B}
\end{aligned} \tag{85}$$

In the numerical example we obtain

$$\begin{aligned}
\pi_1 &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} [(.64 - .0005q_1) q_1 - .28q_1] \\
&= \max_{q_1} [.64q_1 - .0005q_1^2 - .28q_1] \\
\Rightarrow \frac{d\pi}{dq_1} &= .64 - .001q_1 - .28 = 0 \\
\Rightarrow .001q_1 &= .36 \\
\Rightarrow q_1 &= 360
\end{aligned} \tag{86}$$

The optimum for firm two is given by

$$\begin{aligned}
q_2^*(p) &= \frac{A-c}{2B} - \frac{q_1}{2} \\
&= \frac{A-c}{2B} - 0.5 \left(\frac{A-c}{2B} \right) \\
&= \frac{2(A-c)}{4B} - \left(\frac{A-c}{4B} \right) \\
&= \frac{2(A-c)}{4B} - \frac{(A-c)}{4B} \\
&= \frac{(A-c)}{4B} \\
&= \frac{(1-.28)}{.004} = 180
\end{aligned} \tag{87}$$

Total industry supply is given by adding the supply for Firm 1 and the supply for Firm 2.

$$\begin{aligned}
Q^* &= q_1^*(p) + q_2^*(p) \\
&= \frac{(A - c)}{2B} + \frac{(A - c)}{4B} \\
&= \frac{3(A - c)}{4B}
\end{aligned} \tag{88}$$

This is larger than the Cournot equilibrium output of $\frac{2(A-c)}{3B}$. The price is given by

$$\begin{aligned}
p^* &= A - BQ^* \\
&= A - B \left(\frac{3(A - c)}{4B} \right) \\
&= A - \frac{3(A - c)}{4} \\
&= \frac{1}{4}A + \frac{3}{4}c \\
&= \frac{A + 3c}{4}
\end{aligned} \tag{89}$$

This Stackelberg price is lower than the Cournot price. The example industry supply is given by $Q = q_1 + q_2 = 360 + 180 = 540$ with an industry price of $p = 1000 - .001(540) = 0.46$. This is lower than the Cournot price but still higher than the competitive price. Consumers are thus better off under the Stackelberg model.

In the Stackelberg model, the first mover produces more output and has higher profits than in the Cournot case. Note that even though the second firm has full information about the decision of the first firm, it is worse off than in the Cournot case when it doesn't know the actual response of the first firm. What makes firm two worse off is that the decision of the first firm is irreversible in the sense that it is fully committed to $\frac{A-c}{2B}$. In a Cournot model the first firm would not be fully committed since this is not the best response to $q_2 = \frac{A-c}{4B}$. If firm two said it was going to produce $\frac{A-c}{4B}$, Firm 1 would produce

$$\begin{aligned}
q_1^* &= \frac{A - c}{2B} - \frac{q_2}{2} \\
&= \frac{A - c}{2B} - \frac{A - c}{8B} \\
&= \frac{3(A - c)}{8B}
\end{aligned} \tag{90}$$

and we wouldn't have a Nash equilibrium.

8. COMPARISON OF THE EXAMPLE MODELS

For the general 2 firm duopoly model with linear inverse demand and constant marginal cost we summarize the results in table 2. For the numerical example table 3 summarizes the results.

Graphically we can compare industry prices in Figure 12 where the Bertrand price is the same as the competitive price. The monopoly price is $p^M = \$0.64$, the Cournot price is $p^C R = \$0.52$, The Stackelberg price is $p^S = \$0.46$, and the competitive price is $p^C = MC = \$0.28$.

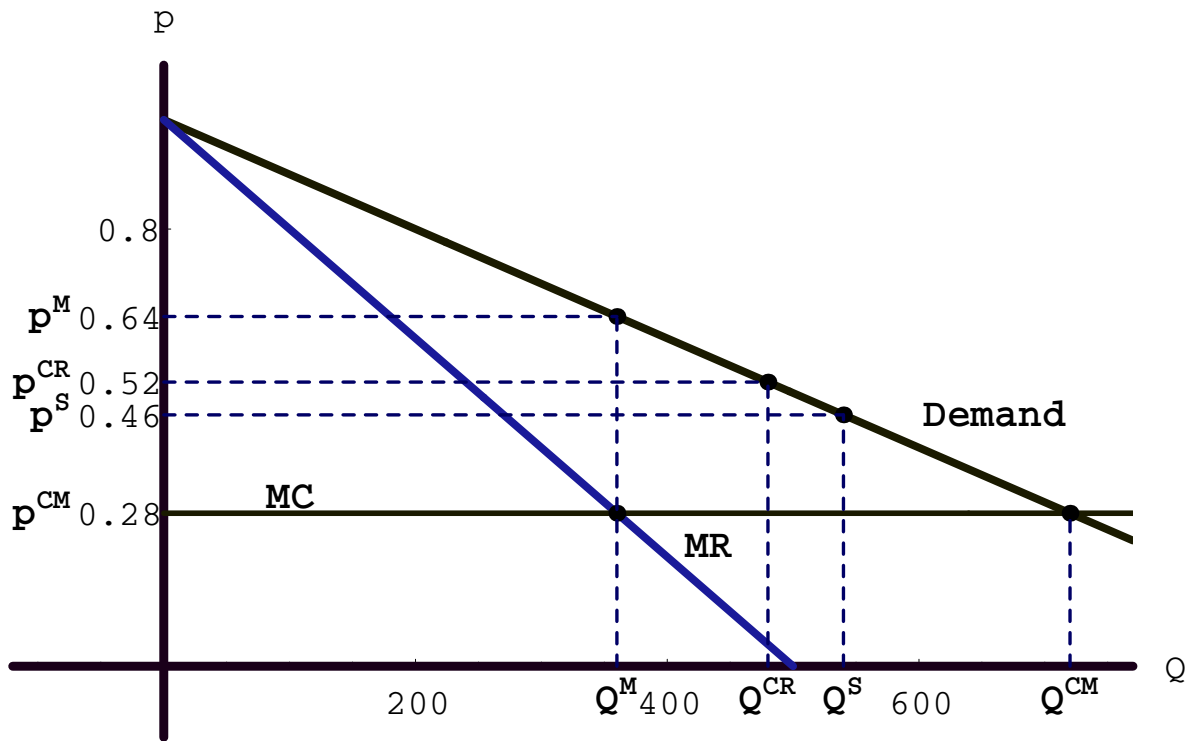
TABLE 2. Optimal Quantity and Price in Duopoly Models

	Firm 1 quantity	Firm 2 quantity	Total quantity	Price
Competition	?	?	$Q = \frac{A-c}{B}$	$p = c$
Cartel/Monopoly	?	?	$Q = \frac{A-c}{2B}$	$p = \frac{A+c}{2}$
Cournot Duopoly	$Q = \frac{(A-c)}{3B}$	$Q = \frac{(A-c)}{3B}$	$Q = \frac{2(A-c)}{3B}$	$p = \frac{A+2c}{3}$
Bertrand Duopoly	?	?	$Q = \frac{A-c}{B}$	$p = c$
Stackelberg Duopoly	$Q = \frac{(A-c)}{2B}$	$Q = \frac{(A-c)}{4B}$	$Q = \frac{3(A-c)}{4B}$	$p = \frac{A+3c}{4}$

TABLE 3. Optimal Quantity and Price in Example Duopoly Models

	Firm 1 quantity	Firm 2 quantity	Total quantity	Price
Competition	?	?	720	.28
Cartel/Monopoly	?	?	360	.64
Cournot Duopoly	240	240	480	.52
Bertrand Duopoly	?	?	720	.28
Stackelberg Duopoly	360	180	540	.46

FIGURE 12. Graphical Summary of Duopoly Models



9. MORE COURNOT AND STACKELBERG MODELS

9.1. **Duopoly Example 1.** Consider two firms in a Cournot duopoly each with constant marginal cost of 0.35 and an industry demand curve of $Q = 1300 - 800p$. If the first firm produces 500 units, we can determine how many units the second firm will want to produce.

$$\begin{aligned}
q_2 &= 1300 - 800p - 500 \\
&= 800 - 800p \\
\Rightarrow 800p &= 800 - q_2 \\
\Rightarrow p &= 1 - .00125q_2 \\
R_2 &= (1 - .00125q_2)q_2 \\
&= q_2 - .00125q_2^2 \\
MR_2 &= 1 - .0025q_2 = .35 = MC_2 \\
\Rightarrow .65 &= .0025q_2 \\
\Rightarrow q_2 &= \frac{.65}{.0025} = 260 \\
&\text{or} \\
\pi_2 &= (1 - .00125q_2)q_2 - .35q_2 \\
&= q_2 - .00125q_2^2 - .35q_2 \\
&= .65q_2 - .00125q_2^2 \\
\frac{d\pi}{dq_2} &= .65 - .0025q_2 = 0 \\
\Rightarrow q_2 &= \frac{.65}{.0025} = 260
\end{aligned} \tag{91}$$

We can also find the second firm's best response function as follows.

$$\begin{aligned}
q_2 &= 1300 - 800p - q_1 \\
\Rightarrow 800p &= 1300 - (q_1 + q_2) \\
\Rightarrow p &= 1.625 - .00125(q_1 + q_2) \\
\pi_2 &= (1.625 - .00125(q_1 + q_2))q_2 - .35q_2 \\
&= 1.625q_2 - .00125q_1q_2 - .00125q_2^2 - .35q_2 \\
&= 1.275q_2 - .00125q_1q_2 - .00125q_2^2 \\
\frac{d\pi}{dq_2} &= 1.275 - .00125q_1 - .0025q_2 = 0 \\
\Rightarrow q_2 &= \frac{1.275}{.0025} - \frac{.00125}{.0025}q_1 \\
&= 510 - .5q_1
\end{aligned} \tag{92}$$

We can also find the first firm's best response function.

$$\begin{aligned}
q_1 &= 1300 - 800p - q_2 \\
\Rightarrow 800p &= 1300 - (q_1 + q_2) \\
\Rightarrow p &= 1.625 - .00125(q_1 + q_2) \\
\pi_1 &= (1.625 - .00125(q_1 + q_2))q_1 - .35q_1 \\
&= 1.625q_1 - .00125q_1^2 - .00125q_1q_2 - .35q_1 \\
&= 1.275q_1 - .00125q_1^2 - .00125q_1q_2 \\
\frac{d\pi}{dq_1} &= 1.275 - .0025q_1 - .0125q_2 = 0 \\
\Rightarrow q_1 &= \frac{1.275}{.0025} - \frac{.00125}{.0025}q_2 \\
&= 510 - .5q_2
\end{aligned} \tag{93}$$

The equilibrium for this market is found by substituting one firm's best response function into the other firm's response function.

$$\begin{aligned}
q_2 &= 510 - .5q_1 \\
&= 510 - .5(510 - .5q_2) \\
&= 510 - 255 - .25q_2 \\
&= 255 - .25q_2 \\
\Rightarrow .75q_2 &= 255 \\
\Rightarrow q_2 &= 340 \\
q_1 &= 510 - .5q_2 \\
&= 510 - .5(340) \\
&= 510 - 170 \\
&= 340
\end{aligned} \tag{94}$$

9.2. Duopoly Example 2. Consider two firms in a Cournot duopoly each with constant marginal cost of 1.6 and an industry demand curve of $Q = 1700 - 500p$. If the first firm produces 500 units, how many will the second firm want to produce?

$$\begin{aligned}
q_2 &= 1700 - 500p - 500 \\
&= 1200 - 500p \\
\Rightarrow 500p &= 1200 - q_2 \\
\Rightarrow p &= 2.4 - .002q_2 \\
R_2 &= (2.4 - .002q_2)q_2 \\
&= 2.4q_2 - .002q_2^2 \\
MR_2 &= 2.4 - .004q_2 = 1.6 = MC_2 \\
\Rightarrow .8 &= .004q_2 \\
\Rightarrow q_2 &= \frac{.8}{.004} = 200 \\
&\text{or} \\
\pi_2 &= (2.4 - .002q_2)q_2 - 1.60q_2 \\
&= 2.4q_2 - .002q_2^2 - 1.60q_2 \\
&= .8q_2 - .002q_2^2 \\
\frac{d\pi}{dq_2} &= .80 - .004q_2 = 0 \\
\Rightarrow q_2 &= \frac{.8}{.004} = 200
\end{aligned} \tag{95}$$

Show the second firm's best response function.

$$\begin{aligned}
q_2 &= 1700 - 500p - q_1 \\
\Rightarrow 500p &= 1700 - (q_1 + q_2) \\
\Rightarrow p &= 3.4 - .002(q_1 + q_2) \\
\pi_2 &= (3.4 - .002(q_1 + q_2))q_2 - 1.60q_2 \\
&= 3.4q_2 - .002q_1q_2 - .002q_2^2 - 1.60q_2 \\
&= 1.80q_2 - .002q_1q_2 - .002q_2^2 \\
\frac{d\pi}{dq_2} &= 1.80 - .002q_1 - .004q_2 = 0 \\
\Rightarrow q_2 &= \frac{1.8}{.004} - \frac{.002}{.004}q_1 \\
&= 450 - .5q_1
\end{aligned} \tag{96}$$

Show the first firm's best response function.

$$\begin{aligned}
q_1 &= 1700 - 500p - q_2 \\
\Rightarrow 500p &= 1700 - (q_1 + q_2) \\
\Rightarrow p &= 3.4 - .002(q_1 + q_2) \\
\pi_1 &= (3.4 - .002(q_1 + q_2))q_1 - 1.6q_1 \\
&= 3.4q_1 - .002q_1^2 - .002q_1q_2 - 1.6q_1 \\
&= 1.8q_1 - .002q_1^2 - .002q_1q_2 \\
\frac{d\pi}{dq_1} &= 1.8 - .004q_1 - .002q_2 = 0 \\
\Rightarrow q_1 &= \frac{1.8}{.004} - \frac{.002}{.004}q_2 \\
&= 450 - .5q_2
\end{aligned} \tag{97}$$

What is the equilibrium for this market?

$$\begin{aligned}
q_2 &= 450 - .5q_1 \\
&= 450 - .5(450 - .5q_2) \\
&= 450 - 225 - .25q_2 \\
&= 225 - .25q_2 \\
\Rightarrow .75q_2 &= 225 \\
\Rightarrow q_2 &= 300 \\
q_1 &= 450 - .5q_2 \\
&= 450 - .5(300) \\
&= 450 - 150 \\
&= 300
\end{aligned} \tag{98}$$

9.3. Stackelberg Example. Consider two firms in a Stackelberg duopoly each with constant marginal cost of .2 and an industry demand curve of $Q = 1100 - 400p$. If the first firm produces 200 units and the second firm is a follower, we can find the output of the second firm by maximizing profits given the first firm's production of 200.

$$\begin{aligned}
q_2 &= 1100 - 400p - 200 \\
&= 900 - 400p \\
\Rightarrow 400p &= 900 - q_2 \\
\Rightarrow p &= 2.25 - .0025q_2 \\
\pi_2 &= (2.25 - .0025q_2)q_2 - .2q_2 \\
&= 2.25q_2 - .0025q_2^2 - .2q_2 \\
&= 2.05q_2 - .0025q_2^2 \\
\frac{d\pi}{dq_2} &= 2.05 - .005q_2 = 0 \\
\Rightarrow q_2 &= \frac{2.05}{.005} = 410
\end{aligned} \tag{99}$$

In general we can find the second firm's best response function using residual demand as follows

$$\begin{aligned}
q_2 &= 1100 - 400p - q_1 \\
\Rightarrow 400p &= 1100 - (q_1 + q_2) \\
\Rightarrow p &= 2.75 - .0025(q_1 + q_2) \\
\pi_2 &= (2.75 - .0025(q_1 + q_2))q_2 - .2q_2 \\
&= 2.75q_2 - .0025q_1q_2 - .0025q_2^2 - .2q_2 \\
&= 2.55q_2 - .0025q_1q_2 - .0025q_2^2 \\
\frac{d\pi}{dq_2} &= 2.55 - .0025q_1 - .005q_2 = 0 \\
\Rightarrow q_2 &= \frac{2.55}{.005} - \frac{.0025}{.005}q_1 \\
&= 510 - .5q_1
\end{aligned} \tag{100}$$

If the first firm is a Stackelberg leader and uses the second firm's best response function to obtain its residual demand, we can find its optimal output by maximizing profit subject to the response of Firm 2 as follows.

The residual demand facing Firm 1 is given by $q_1(p) = 1100 - 400p - q_2(p)$. This then gives

$$\begin{aligned}
q_1(p) &= 1100 - 400p - q_2(p) \\
&= 1100 - 400p - [510 - .5q_1] \\
&= 590 - 400p + .5q_1 \\
\Rightarrow .5q_1(p) &= 590 - 400p \\
\Rightarrow q_1(p) &= 1180 - 800p \\
\Rightarrow 800p &= 1180 - q_1 \\
\Rightarrow p &= 1.475 - .00125q_1
\end{aligned} \tag{101}$$

Thus the residual inverse demand curve is given by $p = 1.475 - .00125q_1$. The profit maximization problem is given by

$$\begin{aligned}
\pi &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} [(1.475 - .00125q_1) q_1 - .2q_1] \\
&= \max_{q_1} [1.475q_1 - .00125q_1^2 - .2q_1] \\
\Rightarrow \frac{d\pi}{dq_1} &= 1.475 - .0025q_1 - .2 = 0 \\
\Rightarrow .0025q_1 &= 1.275 \\
\Rightarrow q_1 &= 510 \\
\pi &= \max_{q_1} [p q_1 - C(q_1)] \\
&= \max_{q_1} [(1.475 - .00125q_1) q_1 - .2q_1] \\
&= \max_{q_1} [1.475q_1 - .00125q_1^2 - .2q_1] \\
\Rightarrow \frac{d\pi}{dq_1} &= 1.475 - .0025q_1 - .2 = 0 \\
\Rightarrow .0025q_1 &= 1.275 \\
\Rightarrow q_1 &= 510
\end{aligned} \tag{102}$$

The optimum for firm two is given by

$$\begin{aligned}
q_2(p) &= 510 - .5q_1 \\
&= 510 - .5(510) \\
&= 255
\end{aligned} \tag{104}$$

10. MULTIPERIOD GAMES AND OLIGOPOLY

10.1. The idea of a multi-period game. The models developed to this point have assumed a one period structure where the game is only played once. In the real world most firms engage in competition over a long time period. For example, two supermarkets in the same town compete over many products and over

months and years. They will take into account how the other has behaved in the past in making decisions. The firms may also signal to each other that they are willing to implicitly cooperate or they can implicitly threaten to punish a firm which does not cooperate.

10.2. A single period game for extension to multiple periods. Consider a simple single period game based on the previous duopoly problem. Suppose instead of an infinite number of output levels each firm is constrained to choose either 180 or 240 units of output. The Cournot solution is to produce 240 each while the cartel solution is to produce 180 each for a total of 360. Suppose the firms cannot explicitly collude. Based on the information in the table created previously we can compute the profits to each firm for each of its strategies for each of the other firms strategies. For example if the first firm produces 240 and the second firm produces 180, the first firm has profits of 72. This is computed using the residual demand equation $q_1 = 1000 - 1000p - q_2$ which gives residual inverse demand of $p = 1 - .001(q_1 + q_2)$. The profits are computed from

$$\begin{aligned}\pi_1 &= pq_1 - C(q) \\ &= (1 - .001(q_1 + q_2))q_1 - .28q_1 \\ &= q_1 - .001q_1^2 - .001q_1q_2 - .28q_1\end{aligned}\tag{105}$$

For $q_1 = 240$ and $q_2 = 180$ this will give

$$\begin{aligned}\pi_1 &= q_1 - .001q_1^2 - .001q_1q_2 - .28q_1 \\ &= 240 - .001(240)^2 - .001(240)(180) - .28(240) \\ &= 72\end{aligned}\tag{106}$$

Similarly if $q_1 = 240$ and $q_2 = 240$ the first firm has profits of 57.6. We can summarize in a table as follows where the first number in each cell is the payoff to Firm 1.

TABLE 4. Optimal Quantity and Price in Duopoly Models

	Firm 2			
Firm 1	240		180	
240	57.6	57.6	72	54
180	54	72	64.8	64.8

The question then is what is the optimal strategy for each firm. Since the firms do not know which strategy the other will play they must make a decision ex ante. The way to play the game is to consider the best strategy contingent on the other firm's action. Consider the case for Firm 1. Suppose Firm 2 chooses to produce 180, then Firm 1 will have profits of \$72 by producing 240 and profits of \$64.8 by producing 180. So Firm 1 will prefer 240. Suppose instead that Firm 2 chooses to produce 240. Firm 1 will have profits of \$57.6 by choosing 240 and \$54 by choosing 180. Thus Firm 1 will prefer 240. Because Firm 1 will prefer 240 in either case, 240 is a dominant strategy. Now consider Firm 2. If Firm 1 chooses 240, then Firm 2 is better off by choosing 240. And if Firm 1 chooses 180, Firm 2 is better off with 240. So Firm 2 has a dominant strategy of 240. Thus the equilibrium of this game is for each firm to produce 240 units which is the duopoly solution. This has total profits of $\$57.6 + \$57.6 = \$115.2$. This is not the best solution for the firms as the cartel solution of 360 total units of product will give profits of \$129.6. This single period game where the firms do not collude results in a lower profit outcome for the firms than is possible if they do collude. This game is a prisoner's dilemma because both firms have dominant strategies that lead to a payoff that is inferior to what it would be if they cooperated.

10.3. An infinitely repeated single period game.

10.3.1. *Signally a willingness to collude.* A firm could signal that it is willing to collude by choosing to produce 180 units for several periods even though it has a lower payoff. The other firm may notice this and voluntarily cut back on output and its profits knowing that the first firm will not keep this up forever. If the second firm doesn't respond, the first firm may return to the higher level of 240.

10.3.2. *Threatening to punish.* If both firms produce 240, they each make \$57.6. If they both produce 180, they each make \$64.8. Suppose Firm 1 makes it known that it will produce 180 as long as Firm 2 does the same. This could be done by announcing a list price of \$.64 and selling no more than 180 units. Assuming the other firm offers only 180 units this price will hold. The first firm could offer discounts if Firm 2 offers more than 180 and the market price starts to fall. Firm 1 could make it very clear that if it has to discount prices at all to remain competitive it will produce 240 units which is the Cournot solution with an equilibrium price of \$.52. If Firm 2 believes Firm 1 will follow this strategy, it will produce 180 units since if it tries to produce 240, Firm 1 will raise its output and lower the profits of Firm 2.

10.4. Issues in repeated games.

10.4.1. *Credibility.* For a threat or signal to be meaningful, it must be credible. A credible threat is one that a firm's rivals believe is rational in the sense that it is in the best interest of the firm to continue to employ it. For example a threat to price below marginal cost forever is not a credible threat.

10.4.2. *Interest rates.* If interest rates are very high, the impact of future periods income is reduced. So situations with lower interest rates may lead to more meaningful threats and signals.

10.4.3. *Length of game.* The longer the game, the more chance that a threat or signal has meaning.

10.4.4. *Perfect Nash equilibria.* Nash equilibria in which threats are credible are called perfect Nash equilibria. For example in a two period game, a threat by Firm 1 to produce more than 240 in the second period if Firm 2 produces more than 180 in the first period is not credible since in the last period of a game the best solution is the one period solution of producing 240.

10.4.5. *Subgame perfect Nash equilibria.* A subgame is a game that starts in period t and continues until the end of the game. If a proposed set of strategies is the best response in any subgame, then these strategies are called a subgame perfect Nash equilibrium. The problem with finite games is that threats are usually not meaningful in the last period and so they are not meaningful in previous period.