

A LITTLE REAL ANALYSIS AND TOPOLOGY

1. NOTATION

Before we begin some notational definitions are useful.

- (1) $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of integers.
- (2) $Q = \{\frac{a}{b} : a \in Z, b \in Z - \{0\}\}$ is the set of rational numbers.
- (3) Numbers that cannot be written as ratios or quotients of integers are called irrational numbers and are denoted, \mathbf{I} . Irrational numbers have decimal expansions that neither terminate nor become periodic. Some familiar irrational numbers are $\sqrt{2}$, π , e , and $\sqrt{3}$. If n is a positive integer which is not a perfect square, then \sqrt{n} is an irrational number.
- (4) \mathfrak{R} or \mathbf{R} , which is the union of the sets of rational and irrational numbers, is the set of real numbers.
- (5) \mathfrak{R}_+ is the set of non-negative real numbers.
- (6) \mathfrak{R}_{++} is the set of positive real numbers.
- (7) \mathbb{C} or \mathbf{C} is the set of the complex numbers. They are an extension of the real numbers by the inclusion of the imaginary unit i , satisfying $i^2 = -1$. Every complex number can be written in the form $x + iy$, where x and y are real numbers called the real part and the imaginary part of the complex number.
- (8) \mathbf{R}^n is the set of all n -dimensional points or vectors, called Euclidean space.
- (9) $R_+^n = \{x : x \in R^n, x \geq 0\}$
- (10) $R_{++}^n = \{x : x \in R^n, x > 0\}$
- (11) $2^{R_+^n} = \{S : S \subseteq R_+^n\}$
- (12) $x \leq y$ iff $x_i \leq y_i, i=1, \dots, n$
- (13) $x \leq y$ iff $x_i \leq y_i$ and $x \neq y$
- (14) $x < y$ iff $x_i < y_i, i=1, \dots, n$
- (15) $x <^* y$ iff $x_i < y_i$ or $x_i = y_i = 0, i=1, \dots, n$

2. INTERVALS

2.1. Open intervals. If a and b are two numbers with $a < b$, then the **open interval** from a to b is the collection of all numbers which are both larger than a and smaller than b . The open interval consists of all numbers between a and b . A compact way of writing this is $a < x < b$. We denote an open interval with parentheses as (a, b) .

2.2. Closed intervals. If a and b are two numbers with $a < b$, then the **closed interval** from a to b is the collection of all numbers which are both greater than or equal to a and less than or equal to b . The closed interval consists of all points between a and b including a and b . A compact way of writing this is $a \leq x \leq b$. We denote a closed interval with brackets as $[a, b]$.

2.3. Half-open intervals. Intervals that are closed on one end and open on the other are called half-open intervals. We can denote these as half open on the right or the left and use a mixture of brackets and parentheses as appropriate.

2.4. Infinite intervals.

$$\begin{aligned}(a, +\infty) &= \{x : x > a\}, & [a, +\infty) &= \{x : x \geq a\} \\ (-\infty, a) &= \{x : x < a\}, & (-\infty, a] &= \{x : x \leq a\}\end{aligned}$$

2.5. **The real line.** The real line \mathbb{R} is sometimes referred to as the interval $(-\infty, +\infty)$.

2.6. **Absolute value.** If x is any real number, the absolute value of x , denoted by $|x|$, is defined as follows,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

2.7. **Distance on the real line.** We define the *distance* between two real numbers x and y is the absolute value of their difference, i.e. $|x - y|$.

3. UPPER BOUNDS, MAXIMAL ELEMENT, LEAST UPPER BOUND (SUPREMUM)

3.1. Definitions.

Definition 1 (Upper bound). Let S be a set of real numbers. If there is a real number b such that $x \leq b$, for every $x \in S$, then b is called an upper bound for S and we say that S is bounded above by b .

We say *an* upper bound because every number greater than b is also an upper bound for S . If an upper bound b is also a member of S , then b is called the *largest member* or *maximal element* of S . There can be at most one b . We write

$$b = \max S$$

A set with no upper bound is said to be *unbounded above*.

Definition 2 (Lower bound). Let S be a set of real numbers. If there is a real number b such that $x \geq b$, for every $x \in S$, then b is called a lower bound for S and we say that S is bounded below by b .

We say *a* lower bound because every number less than b is also a lower bound for S . If a lower bound b is also a member of S , then b is called the *smallest member* or *minimal element* of S . There can be at most one b . We write

$$b = \min S$$

A set with no lower bound is said to be *unbounded below*.

The set $\mathbb{R}^+ = (0, +\infty)$ is unbounded above. It has no upper bound and no maximal element. It is bounded below by 0 , but has no minimal element. The interval $S = [0, 1)$ is bounded above by 1 but has no maximal element. It is bounded below by 0 , which is also its minimal element.

Definition 3 (Least upper bound). Let S be a set of real numbers bounded above. A real number b is called a least upper bound for S if it has following two properties:

- a. b is an upper bound for S .
- b. No number less than b is an upper bound for S .

The least upper bound for a set of real numbers is unique. We usually refer to the least upper bound of a set by the term *supremum* or *sup*. Thus we write $b = \sup S$ if b is the supremum of the set S . If S has a maximal element then $\max S = \sup S$.

Definition 4 (Greatest lower bound). Let S be a set of real numbers bounded below. A real number b is called a greatest lower bound for S if it has following two properties:

- a. b is a lower bound for S .
- b. No number greater than b is a lower bound for S .

The greatest lower bound for a set of real numbers is unique. We usually refer to the greatest lower bound of a set by the term *infimum* or *inf*. Thus we write $b = \inf S$ if b is the infimum of the set S . If S has a minimal element then $\min S = \inf S$.

3.2. Completeness axiom.

Axiom 1 (Completeness). *Every nonempty set of real numbers S which is bounded above has a supremum, that is, there is a real number b such that $b = \sup S$.*

Every nonempty set of real numbers S which is bounded below has an infimum, that is, there is a real number b such that $b = \inf S$.

Theorem 1 (approximation). *Let S be a nonempty set of real numbers with a supremum $b = \sup S$. Then for every $a < b$ there is some x in S such that*

$$a < x \leq b$$

This says that a set with a supremum has members that are arbitrarily close to the supremum.

4. TWO IMPORTANT INEQUALITIES

4.1. **Triangle inequality.** For arbitrary x and y we have

$$|x + y| \leq |x| + |y|$$

4.2. **Cauchy-Schwarz inequality.** If a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n are arbitrary real numbers, we have

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right)$$

Moreover, if some $a_i \neq 0$ equality holds if and only if there is a real number x such that $a_k x + b_k = 0$ for each $k = 1, 2, \dots, n$.

5. SEQUENCES AND LIMITS

5.1. **Sequence.** A sequence is an ordered set of mathematical objects. Sequences of objects are most commonly denoted using braces. For example, the symbol $\{2n\}_{n=1}^{\infty}$ denotes the infinite sequence of even numbers $\{2, 4, \dots, 2n, \dots\}$. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ denotes an infinite sequence that becomes closer and closer to zero. We usually denote a sequence by $\{x_n\}$ or by $\{x_i\}_{i=1}^n$.

5.1.1. *Decreasing sequence.* A sequence $\{a_1, a_2, a_3, \dots\}$ for which $a_1 \geq a_2 \geq a_3 \geq \dots$ is called a decreasing sequence.

5.1.2. *Increasing sequence.* A sequence $\{a_1, a_2, a_3, \dots\}$ for which $a_1 \leq a_2 \leq a_3 \leq \dots$ is called an increasing sequence. Alternatively, for a sequence $\{a_n\}$, if $a_{n+1} - a_n > 0$ for $n \geq x$, then a_n is increasing for $n \geq x$.

5.1.3. *Monotonic sequence.* A sequence $\{a_1, a_2, a_3, \dots\}$ such that either (1) $a_{i+1} \geq a_i$ for every $i \geq 1$ or (2) $a_{i+1} \leq a_i$ for every $i \geq 1$ is called a monotonic sequence.

5.2. Subsequence. A subsequence of a sequence $\{x_i\}_{i=1}^n$ is a derived sequence $\{y_i\}_{i=1}^N = \{x_{i+j}\}$ for some $j \geq 0$ and $N \leq n - j$. More generally, the word subsequence is used to mean a sequence derived from a sequence S by discarding some of its terms. We can write a subsequence in a more precise manner by considering a strictly increasing function k which assigns to each positive integer i another positive integer $k(i)$. Then we can define the composite function $(x \circ k)(i) = x_{k(i)} = x_{k_i}$. This composite function is then a subsequence of $\{x_i\}$. For example if $k(i) = 2i$, then for the sequence $\{x_i\} = \{1, 3, 7, 15, 31, 63, \dots\}$, the subsequence $\{x_{k_i}\}$ is $\{3, 15, 63, \dots\}$.

5.3. Convergent sequence. A sequence $\{x_i\}_{i=1}^n$ converges to the limit point p

$$\lim_{n \rightarrow \infty} \{x_i\}_{i=1}^n = p$$

if, for any $\varepsilon > 0$, there exists an N such that $|x_n - p| < \varepsilon$ for $n > N$. If does not converge, it is said to diverge. We also say that $\{x_n\}$ converges to p and we write $\{x_n\} \rightarrow p$ as $n \rightarrow \infty$ or simply $x_n \rightarrow p$.

5.4. Monotonic convergence. If a sequence $\{x_n\}$ is increasing it converges to a finite limit or to $+\infty$. If a sequence $\{x_n\}$ is decreasing it converges to a finite limit or to $-\infty$. Every bounded monotonic sequence converges.

5.5. Bolzano-Weierstrass principle for the real line. Every bounded sequence $\{x_n\}_{n \geq 1}$ of real numbers possesses a convergent subsequence $\{x_{k_n}\}$.

6. FUNCTIONS

6.1. Functions of a real variable. A function of a real variable x with **domain** D is a rule that assigns a unique real number to each number x in D . Functions are often given letter names such as f , g , F , or ϕ . We often call x the independent variable or the argument of f . If g is a function and x is a number in D , then $g(x)$ denotes the number that g assigns to x . We sometimes make the idea that F has an argument (we substitute a number for the variable in F) explicit by writing $F(\cdot)$. In the case of two variables we sometimes use $y = f(x)$ for the value of f evaluated at the number x . Note the difference between ϕ and $\phi(x)$.

(i) The domain of a function

The domain is the set of all values that can be substituted for x in the function $f(\cdot)$. If a function f is defined using an algebraic formula, we normally adopt the convention that the domain consists of all values of the independent variable for which the function gives a meaningful value (unless the domain is explicitly mentioned).

(ii) The range of a function

Let g be a function with domain D . The set of all values $g(x)$ that the function assumes is called the range of g . To show that a number, say a , is in the range of a function f , we must find a number x such that $f(x) = a$.

6.2. Continuous functions of a real variable.

6.2.1. Continuity at interior points. A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

6.2.2. *Continuity at endpoints.* A function $y = f(x)$ is **continuous at a left endpoint a** of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

A function $y = f(x)$ is **continuous at a right endpoint b** of its domain if

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

If a function is not continuous at a point c , we say that f is discontinuous at c .

6.3. Max-min theorem for continuous functions of a real variable.

Theorem 2. *If f is continuous at every point of a closed interval $[a, b] \subset \mathbb{R}$, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.*

6.4. **Real-valued functions of several variables.** Suppose D is a set of n -tuples of real numbers $(x_1, x_2, x_3, \dots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number

$$y = f(x_1, x_2, x_3, \dots, x_n)$$

to each element in D . The set D is the function's **domain**. The set of y -values taken on by f is the range of the function. The symbol y is the **dependent variable** of f , and f is said to be a function of the n **independent variables** x_1 to x_n . We also call the x 's the function's **input variables** and we call y the function's **output variable**.

A real-valued function of two variables is just a function whose domain is \mathbb{R}^2 and whose range is a subset of \mathbb{R}^1 , or the real numbers. If we view the domain D as column vectors in \mathbb{R}^n , we sometimes write the function as

$$f \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

6.5. **Continuous functions.** Let S be a nonempty set in \mathbb{R}^n . A function $f: S \rightarrow \mathbb{R}^1$ is said to be continuous at $\bar{x} \in S$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in S$ and $\|x - \bar{x}\| < \delta$ imply that $|f(x) - f(\bar{x})| < \varepsilon$.

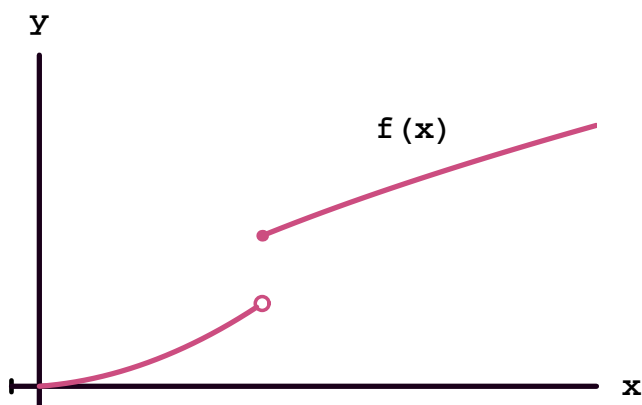
6.6. **Upper semicontinuous functions.** Let S be a nonempty set in \mathbb{R}^n . A function $f: S \rightarrow \mathbb{R}^1$ is said to be upper semi-continuous at $\bar{x} \in S$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in S$ and $\|x - \bar{x}\| < \delta$ imply that $f(x) - f(\bar{x}) < \varepsilon$.

6.7. **Lower semicontinuous functions.** Let S be a nonempty set in \mathbb{R}^n . A function $f: S \rightarrow \mathbb{R}^1$ is said to be lower semi-continuous at $\bar{x} \in S$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in S$ and $\|x - \bar{x}\| < \delta$ imply that $f(x) - f(\bar{x}) > -\varepsilon$. A function which is both upper and lower semi-continuous is said to be continuous.

A second definition is that the function $f(x)$ is upper semi-continuous iff the sets $\{x: f(x) \geq y\}$ are closed for all y in \mathbb{R}^1 while the function $f(x)$ is lower semi-continuous iff the sets $\{x: f(x) \leq y\}$ are closed for all y in \mathbb{R}^1 .

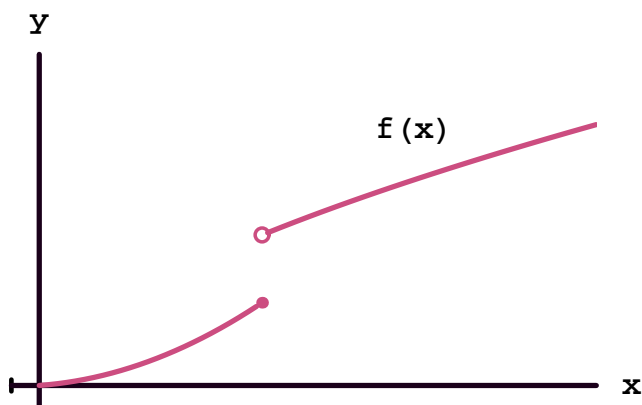
6.8. Intuition and geometrical analysis of a semi-continuous function. The easiest definition of semi-continuity to visualize is the one based on closed sets. Consider a function with a jump at \bar{x} . If the function takes the upper value at \bar{x} and greater, but the lower value for all points to the left of \bar{x} , the function is upper semi-continuous. This is obvious because the set of points x where $f(x)$ is greater than or equal to the value at the jump point is closed but the set where $f(x)$ is less than or equal to the value at the jump point is open. A lower semi-continuous function would take the lower value at the jump point. Consider the function in figure 1.

FIGURE 1. Upper Semi-continuous Function



The function takes the upper value at the jump point as indicated by the solid disk as compared to the open circle. Consider the function in figure 2.

FIGURE 2. Upper Semi-continuous Function



With lower semi-continuity, the function takes the lower value at the jump point.

7. POINT SET TOPOLOGY

7.1. Neighborhood. The neighborhood of a point in \mathbb{R}^n (also called an epsilon-neighborhood) is the set of points inside an n-ball with center x and radius $\varepsilon > 0$. A set containing an open neighborhood is also called a neighborhood. No matter how small ε (as long as it is greater than zero), all points within it are in the neighborhood of x .

7.2. Open set. A set is open if every point in the set has a neighborhood lying in the set. In one-space, the open set is an open interval. In two-space, the open set is a disk. In three-space, the open set is a ball.

7.3. Closed set. A closed set is a set which contains all of its limit points. Therefore, a closed set C is one for which, whatever point x is picked outside of C , x can always be isolated in some open set which doesn't touch C . This can be characterized in a number of different ways..

- a: Sequences in S that converge do so within S .
- b: Every point outside S has a neighborhood disjoint from S .
- c: The complement of S is an open set.

It is possible for a set to be neither open nor closed, e.g., the half-closed interval $(0,1]$. The set $\{(x,y):x^2 + y^2 < 1\}$ is an open set while the set $\{(x,y):x^2 + y^2 \leq 1\}$ is a closed set.

7.4. Limit point. A limit point is a number x such that for all $\varepsilon > 0$, there exists a member, y , of the set S , different from x such that $|y-x| < \varepsilon$. We say that a point $P \in S$ is a limit point of S if every open set around it intersects S . Sometimes limit points are called accumulation points. An **accumulation point** is a point which is the limit of a sequence. If S is a subset of \mathbb{R}^n and $x \in S$, but x is not an accumulation point of S , then x is called an isolated point of S .

7.5. Covering.

7.5.1. Definition of a covering. A collection \mathcal{F} of sets is said to be a covering of a given set S if $S \subseteq \bigcup_{A \in \mathcal{F}} A$. The collection \mathcal{F} is also said to cover S . If \mathcal{F} is a collection of open sets, then \mathcal{F} is called an open covering of S .

7.5.2. Examples of coverings.

- (i) The collection of all intervals of the form $\frac{1}{n} < x < \frac{2}{n}$, ($n = 2, 3, 4, \dots$) is a countable open covering of the interval $0 < x < 1$. To see this write the various terms in the set: $(\frac{1}{2} < x < 1)$, $(\frac{1}{3} < x < \frac{2}{3})$, $(\frac{1}{4} < x < \frac{2}{4})$, $(\frac{1}{5} < x < \frac{2}{5})$, \dots , $(\frac{1}{10000} < x < \frac{2}{10000})$, \dots , $(\frac{1}{n} < x < \frac{2}{n})$.
- (ii) The real line \mathbb{R}^1 is covered by the collection of all open intervals (a,b) . This covering is not countable.
- (iii) The real line \mathbb{R}^1 is covered by the collection of all open intervals of the form $(n, n+2)$ where n is an integer. This covering is countable.
- (iv) Let $S = \{(x, y) : x > 0, y > 0\}$. The collection \mathcal{F} of all circular disks of centers at (x,x) and with radius, where $x > 0$, is a covering of S . This covering is not countable. However, it contains a countable covering of S , namely, all those disks in which x is rational.

7.6. Lindelöf covering theorem. Assume $A \subseteq \mathbb{R}^n$ and let \mathcal{F} be an open covering of A . Then there is a countable subcollection of \mathcal{F} which also covers A .

7.7. Heine-Borel theorem. Let \mathcal{F} be an open covering of a closed and bounded set $A \subseteq \mathbb{R}^n$. Then a finite subcollection of \mathcal{F} also covers A .

7.8. Compact set. A set S in \mathbb{R}^n is said to be compact if, and only if, every open covering of S contains a finite subcover, that is, a finite subcollection which also covers S .

The Heine-Borel theorem then implies that every closed and bounded set in \mathbb{R}^n is compact.

We can summarize these results in a theorem.

Theorem 3. *Let S be a subset of \mathbb{R}^n . Then the following three statements are equivalent.*

- (a) *S is compact.*
- (b) *S is closed and bounded.*
- (c) *Every infinite subset of S has a limit (accumulation) point in S , i.e., from any sequence of elements X_1, X_1, \dots , of S , a subsequence can always be extracted which tends to some limit element X of S .*

7.9. Bolzano-Weierstrass theorem. The **Bolzano-Weierstrass** theorem states that every bounded infinite set in \mathbb{R}^n has an accumulation point. In \mathbb{R}^1 the Bolzano-Weierstrass theorem states that every bounded sequence of real numbers contains a convergent subsequence. The sequence a_1, a_2, a_3, \dots is called bounded if there exists a number L such that the absolute value $|a_n|$ is less than L for every index n . More generally, any sequence a_n in a compact set, has a convergent subsequence.

7.10. Weierstrass extremum theorem. The **Weierstrass** theorem states that a continuous, real-valued function defined over a non-empty compact set attains a maximum and a minimum in the set. Specifically, $f: S \rightarrow \mathbb{R}$ be a continuous real-valued function on a non-empty compact set S . Let $M = \sup_{x \in S} f(x)$ and $m = \inf_{x \in S} f(x)$. Then, there is a point x^M and a point x^m in S such that $f(x^M) = M$ and $f(x^m) = m$, i.e. a continuous function $f(x)$ attains a maximum and a minimum over a compact set.