**CAUTION: Late assignments will not be accepted – no exceptions.

Introductory Exercise on the Basic Optimal Growth Model

Basic Reference:

- [1] (**) “A Simple Illustrative Optimal Growth Model,” posted online at Econ 502 Syllabus Section III.D

Background Materials from Ref. [1]:

As in [1], let $\rho \geq 0$ be an exogenously given discount rate. Let $g > 0$ denote the exogenously given growth rate of labor, and let $\delta \geq 0$ denote the exogenously given capital depreciation rate. Let $\theta \equiv [g + \delta] > 0$. Finally, let sequences of real-valued per-capita capital and consumption levels over $[0, T]$ be denoted by

$$k = (k(t) : t \in [0, T]) ; \quad (1)$$

$$c = (c(t) : t \in [0, T]) . \quad (2)$$

Consider the following intertemporal utility maximization problem:

$$\max_{c,k} \int_0^T u(c(t))e^{-\rho t} dt \quad (3)$$

subject to

$$c(t) = f(k(t)) - \theta k(t) - D_k k(t) , \quad 0 \leq t \leq T ; \quad (4)$$

$$k(0) = k_0 ; \quad (5)$$

$$k(T) = k_T . \quad (6)$$

Let $K$ denote the collection of all twice differentiable functions $k$ taking the form $k:[0, T] \rightarrow R$ with boundary conditions $k(0) = k_0$ and $k(T) = k_T$. Using equation (3) to substitute out for $c(t)$ in the objective function, one obtains a representation for this objective function as a function only of $k$, as follows:

$$J(k) = \int_0^T [u(f(k(t)) - \theta k(t) - D_k k(t))]e^{-\rho t} dt . \quad (7)$$

Problem (3)-(6) then takes the compact form

$$\max_{k \in K} J(k) . \quad (8)$$
THEOREM [20, pp. 10-11]: Let \( \rho \geq 0 \) and \( \theta \equiv [g + \delta] > 0 \) be given. Suppose the utility function \( u : \mathbb{R}_+^+ \to \mathbb{R} \) is twice continuously differentiable with \( u' > 0 \) and \( u'' < 0 \). Suppose the production function \( f : \mathbb{R}_+ \to \mathbb{R} \) is continuous over \( \mathbb{R}_+ \) and twice continuously differentiable with \( f' > 0 \) and \( f'' < 0 \) over \( \mathbb{R}_+ \). Let \( K \) denote the set of all twice differentiable functions \( k \) of the form \( k : [0, T] \to \mathbb{R} \) with \( k(0) = k_0 \) and \( k(T) = k_T \). Then, in order for a function \( k^* \) in \( K \) to be the unique solution for the optimal growth problem (8), it is necessary and sufficient that \( k^* \) solve the following differential system:

\[
Dk^*(t) = f(k^*(t)) - \theta k^*(t) - c^*(t) , \quad t \in [0, T] ; \tag{9}
\]

\[
Dc^*(t) = -\frac{u'(c^*(t))}{u''(c^*(t))} [f'(k^*(t)) - \theta - \rho] , \quad t \in [0, T] . \tag{10}
\]

PART A (5 Points): Provide a careful economic interpretation for the maximization problem (8).

PART B (10 Points): Reference [1] (Section E, Figure 3) develops a phase diagram portrait for the Euler-Lagrange equations (9) and (10) for generally specified utility and production functions \( u(c) \) and \( f(k) \). Redo this phase diagram portrait using the following more precisely specified utility and production functions:

\[
u : \mathbb{R}_+^+ \to \mathbb{R} \quad \text{given by} \quad u(c) = B - \exp(-\beta c) , \quad B > 1 , \quad \beta > 0 \tag{11}
\]

\[
f : \mathbb{R}_+ \to \mathbb{R} \quad \text{given by} \quad f(k) = k^\alpha , \quad 0 < \alpha < 1 . \tag{12}
\]

Be sure to show your work, step by step, justifying carefully each step. Also, be sure your resulting phase diagram portrait is carefully labeled and carefully explained.

PART C (5 Points): What additional dynamic properties (if any) can be deduced from your phase diagram portrait in Part B in comparison to the general phase diagram portrait developed in [1] (Section E, Figure 3)? Explain carefully.