Long-Run Effects of Differential Savings Rates in a Simple Two-Sector Growth Model

Background Materials:

- “An Illustrative Two-Sector Neoclassical Growth Model”, Packet 18;
- “A Simple Illustrative Optimal Growth Model”, Packet 19 (sections focusing on phase diagram techniques for dynamic growth models with two-dimensional state vectors)

Consider the following economy, assumed to exist over times $t \geq 0$. The economy has an aggregate production function

$$ Y = F(K), $$

where $Y$ denotes aggregate real income (output) and $K$ denotes capital input. The production function $F(K)$ is a twice continuously differentiable function of $K$ satisfying

$$ F(0) = 0, \quad F'(K) > 0, \quad \text{and} \quad F''(K) < 0 \quad \text{for all} \quad K > 0; $$

$$ F'(K) \rightarrow \infty \quad \text{as} \quad K \rightarrow 0; $$

$$ F'(K) \rightarrow 0 \quad \text{as} \quad K \rightarrow \infty. $$

The economy consists of two classes of capital owners holding capital stocks $K_1(t)$ and $K_2(t)$, respectively, at each time $t$, so that the total capital stock at time $t$ is given by

$$ K(t) = K_1(t) + K_2(t). $$

These capital owners receive real incomes $Y_1(t)$ and $Y_2(t)$ at time $t$ in proportion to their capital ownership, i.e.,
Given previous assumptions, this implies that the total real income $Y(t)$ at each time $t$ satisfies $Y(t) = Y_1(t) + Y_2(t)$.

The two classes of capital owners plow back constant fractions $s_1$ (satisfying $0 < s_1 < 1$) and $s_2$ (satisfying $0 < s_2 < 1$) of their real incomes into new capital, which depreciates at a constant rate $\delta > 0$. Thus, the net investment levels of these two classes of capital owners are as follows:

$$D_+ K_1(t) = s_1 Y_1(t) - \delta K_1(t) ;$$
$$D_+ K_2(t) = s_2 Y_2(t) - \delta K_2(t) .$$

Suppose also that the capital holdings of the two classes of capital owners at the initial time 0, $K_1(0)$ and $K_2(0)$, are nonnegative exogenously given amounts satisfying $K_1(0) + K_2(0) > 0$. Finally, suppose that the savings rates $s_1$ and $s_2$ satisfy $s_1 > s_2$.

**Part A: (3 Points)** Formulate the entire model described above as a system of equations with a two-dimensional state vector $(K_1(t), K_2(t))$ at each time $t \geq 0$ and a specified classification of variables with admissibility conditions. **Caution:** Use the equations and assumptions described above – do not modify these equations and assumptions and do not add new equations or new assumptions.

**Part B: (2 Points)** Determine the basic causal system (BCS) for your model in Part A. Show your derivation steps, and justify that what you have derived is indeed the BCS for your Part A model (i.e., explain how it satisfies the definition of a BCS for your Part A model).

**Part C: (4 Points)** Carefully establish that your model in part A has two distinct admissible stationary solutions. Characterize these stationary solutions as precisely as you can, using diagrams for illustration. **Important Note:** Recall that $s_1$ is assumed to be larger than $s_2$.

**Part D: (4 Points)** Using graphical phase diagram techniques, carefully examine the stability properties of these two admissible stationary solutions you established in Part C. In particular, determine what happens to the capital holdings (hence the income and savings) of the two classes of capital owners “in the long run.” Carefully justify all of your assertions.

**Part E: (2 Points)** Provide an economic interpretation for your findings in Part D.