**CAUTION:** Late assignments will not be accepted – no exceptions.

Note 1: Please make an EXTRA copy of your exercise to bring to class on the due date for use in class discussion after you turn in your exercise.

Note 2: Students are permitted to work together in study groups on this exercise, but each student is asked to separately prepare and turn in their exercise answer. The techniques covered in this exercise are essential tools for subsequent exercises and exams, so “free riding” on other people’s work should definitely be avoided!

Background Materials:
- “A Simple Illustrative Optimal Growth Model” (Packet 19);
- Exploring the Dynamics of a Modified Optimal Growth Model

Consider the equations and classification of variables for Model $M^o$, attached to the end of this exercise. Model $M^o$ is a variant of the simple illustrative optimal growth model presented in Packet 19 (page 4). In Model $M^o$, the social planner is assumed to maximize the discounted sum of total utility $u(c(t))L(t)$ for all working agents in the economy over times $t \in [0,T]$ rather than the discounted sum of utility $u(c(t))$ for single “representative” working agents in the economy over times $t \in [0,T]$.

**PART A:** [4 Points] Using the findings in Packet 19 (no need to reprove them), express in analytical form the Euler-Lagrange equations for Model $M^o$, that is, the equations for $Dk(t)$ and $Dc(t)$ for Model $M^o$ that are analogous to the Euler-Lagrange equations (24) and (25) derived for the simple illustrative optimal growth model in Packet 19.

**IMPORTANT HINTS FOR PART A:** For any scalar $x$ and $y$, $e^{x+y} = e^x \cdot e^y$. Also, given any real number $b$, the solution of the scalar differential system $Dx(t) = bx(t), t \geq 0$, is given by $x(t) = e^{bt}x(0), t \geq 0$.

**PART B:** [8 Points] Using your results from Part A, derive a carefully labeled phase diagram for Model $M^o$, analogous to the phase diagram depicted for the simple illustrative optimal growth model in Figure 3 of Packet 19. Be sure to carefully explain and justify the steps you have taken to obtain this phase diagram for Model $M^o$, so that the form of this phase diagram is clearly seen to reflect the dynamic properties of possible solutions for Model $M^o$ (not some other model).

**PART C:** [8 Points total] Suppose the growth rate $g$ for the labor force $L(t)$ in Model $M^o$ is decreased to a smaller value $g^\#$ satisfying $0 < g^\# < g$.

C.1 [4 Points] Carefully show how your phase diagram in Part B for Model $M^o$ is affected by this change in $g$. Justify your assertions with care.

C.2 [4 Points] Provide a careful economic explanation for your findings in Part C.1.
MODEL M\textsuperscript{o}: A MODIFIED OPTIMAL GROWTH MODEL

MODEL M\textsuperscript{o} EQUATIONS:

\[
\max_{c,k} \int_0^T u(c(t))L(t)e^{-\rho t}dt \tag{1}
\]

subject to

\[
L(t) = L_0 e^{gt}, \, 0 \leq t \leq T; \tag{2}
\]
\[
c(t) = f(k(t)) - gk(t) - Dk(t), \, 0 \leq t \leq T; \tag{3}
\]

where \(K\) denotes the collection of all continuously differentiable functions \(k:[0,T] \to R_+\) satisfying \(k(0) = k_0\) and \(k(T) = k_T\)

MODEL M\textsuperscript{o} CLASSIFICATION OF VARIABLES:

Endogenous Variables:

\[L = (L(t) : 0 \leq t \leq T)\] \tag{4}
\[k = (k(t) : 0 \leq t \leq T)\] \tag{5}
\[c = (c(t) : 0 \leq t \leq T)\] \tag{6}

Admissible Exogenous Variables:

\(T = \) End of social planning interval, \(L_0 = \) Labor force at time 0, \(k_0 = \) Initial capital-labor ratio at time 0, \(k_T = \) Target capital-labor ratio for time \(T\), \(g = \) Growth rate of the labor force, and \(\rho = \) Social planning discount rate, where these exogenous variables are required to satisfy the admissibility conditions \(0 < T\), \(0 < L_0\), \(0 < k_0 < k_T\), and \(0 < g < \rho\).

Admissible Exogenous Functional Forms:

- \(u:R_+ \to R_+\) is a twice continuously differentiable function that satisfies \(u'(c) > 0\) and \(u''(c) < 0\) for all \(c \geq 0\). It measures the instantaneous utility of consumption for a representative one-period-lived worker at time \(t\).

- \(f:R_+ \to R_+\) is a production function that satisfies the Standard Neoclassical Production Function Assumptions in Per-Capita Form: namely,

\[
f(0) = 0
\]
\(f(k)\) is continuous over \(R_+ = \{k \in R \mid k \geq 0\}\)
\(f(k)\) is twice continuously differentiable with \(f'(k) > 0\) and \(f''(k) < 0\) over \(R_{++} = \{k \in R \mid k > 0\}\)
(\text{Inada Conditions}) \(\lim_{k \to 0} f'(k) = 0\), and \(\lim_{k \to 0} f'(k) = \infty\)