Applied Numerical Methods in Economics

Homework #2
Electronic answers due on June 21 at 2:00 PM

1. **Gaussian Elimination with Partial Pivoting**

Create a function m-file `myGE` that takes a square matrix as input and that performs the LU decomposition returning the matrices $L$ and $U$ as output. The function must implement the following algorithm to perform the decomposition:

i) Begin loop $(i = 1$ to $n - 1)$

ii) Find the largest entry (in absolute value) in column $i$ from row $i$ to row $n$. If the largest value is zero, signal that a unique solution does not exist and stop.

iii) If necessary, perform a row interchange to bring the value from step 2 into pivot position $(i, i)$.

iv) For $j = i + 1$ to $j = n$, perform $R_j ← R_j - m_{ji}R_i$ where $m_{ji} = a_{ji}/a_{ii}$

v) End loop

vi) If the $(n, n)$ entry is zero, signal that a unique solution does not exist and stop.

Otherwise, solve the solution $x$ by backward substitution.

2. **Cholesky Decomposition**

Create a function m-file `myCD` that takes a square matrix as input and that performs a Cholesky decomposition returning an upper triangular matrix $U$ as output. The function must perform the decomposition by exploiting the following recurrence relations. For the $i$th row:

$$u_{ii} = \left( a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2 \right)^{1/2}; \quad u_{ij} = a_{ij} - \sum_{k=1}^{i-1} \frac{u_{ki}u_{kj}}{u_{ii}}$$

for $j = i + 1, \ldots, n$

3. **Condition Numbers and the Vandermonde Matrix**

A matrix that we are going to study later and that is inherently ill-conditioned is the Vandermonde matrix. For scalars $x_1, x_2, \ldots, x_n$, the Vandermonde matrix has the following form:
Determine the condition number of the Vandermonde matrix based on the spectral norm and the Frobenius norm. Use \( n = 1, 2, 3, 4, \ldots, n_m \), and small natural numbers for \( x_1, x_2, \ldots, x_n \). Here you have to choose the \( x \)'s and a relatively large (natural) number \( m \), and you have to report a table of results; the table will show for each value of \( n \), the condition numbers of the corresponding Vandermonde matrix obtained using the two aforementioned norms.

4. **Ordinary Least Squares with Iterative Algorithms**

Try to solve the “Ordinary Least Square” exercise given in Homework #1 using Gauss Seidel and/or Gauss-Jacobi algorithm noticing that the normal equation used in the estimation is \( X^T X \beta = X^T y \). If you cannot find the solution by using the aforementioned methods, explain why you think that happens.

5. **Newton-Raphson Method**

Create a script m-file that uses the Newton-Raphson method to estimate the root of 
\[
f(x) = e^{-x} - x
\]
start with an initial guess of \( x^{(0)} = 0 \). Make a plot to show the first 4 iterations. The script must be self contained: you must not call any function and all operations needed to find the solution must be part of the script. Create a table showing the value of the estimate at each iteration and the estimated relative error of approximation \( \rho^{(k+1)} = \frac{|x^{(k+1)} - x^{(k)}|}{|x^{(k+1)}|} \). The script must stop iterating when \( \rho < 1E^{-8} \).


Find the point of intersection between the circle \( x^2 + y^2 = 2 \) and the ellipse \( \frac{1}{3} x^2 + \frac{1}{2} y^2 = 2 \) using Newton’s method.

7. **Measures of Risk Preference and Bisection Method**

Consider the problem of an agent who has to decide how much risk coverage to buy from an insurance company. To fix ideas, think about insurance on car accidents. The agent’s wealth is \( w \); if he is lucky (i.e. he has no accident), he keeps this wealth \( w \). If he is unlucky, he has a wealth loss equal to \( d \). Compactly, the payoffs of the agent are \((w, w - d)\). Insurance companies offer to cover the loss, if it materializes, in exchange for a premium. Under full insurance, insurance companies pay \( d \) in the accident state and 0 otherwise. This coverage costs a premium equal to \( \mu \) so the
payoffs under full insurance are \((w - \mu, w - \mu)\). If the agent chooses a coverage \(c\) (say 75\%), the agent receives \(cd\) in the accident state and 0 in the other state. In this case, the premium is \(c\mu\).

Suppose that the probability of having an accident is \(\pi\) and suppose that the agent has a von Neumann-Morgenstern utility with utility index given by \(u(x) = \frac{x^{\sigma} - 1}{1 - \sigma}\) where \(\sigma > 1\) represents his degree of risk aversion. The agent’s problem can be summarized as:

\[
\max_c \{(1 - \pi)u(w - c\mu) + \pi(w - c\mu - (1 - c)d)\}
\]

and the first order condition of optimality of this problem is:

\[
\frac{1 - \pi}{\pi} \frac{u'(w - c\mu)}{u'(w - c\mu - (1 - c)d)} = \frac{d - \mu}{\mu}
\]

We can link the expected loss to the cost of the insurance through:

\[
\mu = (1 + m)\pi d
\]

where \(m \geq 0\) is called the gross markup. If \(m = 0\), we say the insurance premium is statistically or actuarially fair. Combining the first-order condition with the definition of the markup:

\[
\frac{1 - \pi}{\pi} \frac{u'(w - c\mu)}{u'(w - c\mu - (1 - c)d)} = \frac{1 - (1 + m)\pi}{(1 + m)\pi}
\]

Consider \(w = 10\) and values of \(\sigma\) between 1.001 and 5. Propose reasonable values for the other parameters discussed above. Create an script file that uses the Bisection method in conjunction with the last equation to show the following:

(a) A risk-averse agents demand full coverage (i.e. \(c = 1\) if and only if the premium is statistically fair.

(b) For any premium greater than the fair one, the agent demands less than full insurance.

(c) Find the value of \(m\), say \(\hat{m}\), for which the agent does not buy any insurance, i.e. \(c = 0\).

(d) Link \(\hat{m}\) to the value of \(\sigma\). A plot could be illustrative here.

(e) What happens to \(\hat{m}\) when the expected damage \((\pi d)\) diminishes?

(f) When \(d\) converges to zero, which of \(\hat{m}\) and \(\pi d\) diminishes more rapidly. Interpret your result.
8. **Measures of Risk Preference and Inverse Linear Interpolation Method**

Repeat the solution to items (a) and (b) in question 6 by using the Inverse Linear Interpolation Method.