ECONOMICS 533
SPRING 2005
HALLAM EXAM

Problem 1. Go through the tutorial on Multiple Variable Optimization. Provide your own examples and code for the following sections.

a. Part 3
b. Part 4
c. Part 5
d. Part 7

Problem 2. We can represent the probability distribution of random variables with a density or joint density function. A density function is a continuous analog of the probability function for discrete random variables. Consider the following joint density function of two random variables X and Y

$$
\begin{aligned}
f(X, Y) &= e^{-X-Y}, \quad 0 \le X \le \infty, \quad 0 \le Y \le \infty \\
&= 0, \quad \text{otherwise}
\end{aligned}
$$

The bounds imply that the random variables X and Y are between zero and ∞. A proper density function will integrate to one over the range of both variables.

Provide Mathematica code to find the solutions to the following problems.

a. Show that this is a proper density function by showing that

$$\int_0^\infty \int_0^\infty e^{-x-y} = 1$$

using the Integrate command in Mathematica. Define the function as f [x_, y_]:= Exp[-x - y];.

b. We can find the marginal density of one of the random variables by integrating the joint density function over the range of the other random variable. To find the marginal density of X, we integrate the joint density over the range of Y. Find the marginal density of X.

c. Find the marginal density of Y.

d. If bounds of both random variables are fixed constants and the joint density is the product of the marginal densities, then the two variables are independent. Are X and Y independent?

e. We can find the expected value of a jointly distributed random variable by multiplying the joint density by the random variable and then integrating over the range of both variables. Show that the expected value of X and the expected value of Y are both equal to 1.

Date: April 28, 2005.

f. We can find the expected value of a jointly distributed random variable by multiplying the marignal density of a random variable by that random variable and then integrating over the range of that variable. Show that the expected value of X and the expected value of Y are both equal to 1 using this approach.

g. We can find the expected value of the square of a jointly distributed random variable by multiplying the joint density of the random variables by the random variable squared and then integrating over the range of both variables. Show that the expected value of X^2 and the expected value of Y^2 are both equal to 2.

g. The variance of a random variable is given by the expected value of the random variable squared minus the square of the expected value, i.e., Variance[X] = $E[X^2] - E[X]^2$. Show that the variance of both X and Y are both 1.

h. The formula for the covariance of two random variables is E[XY] - E[X]E[Y]. Show that this is zero for these two random variables. Is this consistent with part d?

Problem 3. Consider a portfolio problem. The investor has a negative exponential utility function which is consistent with maximizing a linear function of the mean and variance of the distribution if the variable is normally distributed. If the data is not normally distributed, the use of the mean variance rule may give significantly different proportions in the portfolio. We considered a simple problem in class with one risky and one riskless asset. Let returns to the investor be given by Returns = (1-q)r1 + qR_2 where R_2 is a random variable, and q is the proportion of the portfolio that is placed in the risky asset. This proportion may be larger than one or less than zero. Assume this random variable has a mean of 0.15 and a standard deviation of 0.07 or a variance of 0.049. Also assume that r1 = 0.03. The investor has a negative exponential utility function given by

$$\begin{aligned} u(x) &= 1 - e^{-a\,x} \\ &= 1 - e^{-30\,x} \end{aligned}$$

a. What is the expected return to this portfolio?

b. Assume for the moment, the investor's objective function is given by

$$u(Returns) = E(Returns) - \frac{30}{2} Var(Returns)$$

What is an expression for u(Returns) as a function of q?

c. By taking the derivative of u(Returns) with respect to q, find a formula for the optimal amount to invest in the risky asset. Use Mathematica to derive the formula.

d. Using the values a = 30, r1 = 0.03, $\mu 2$ = 0.15, and $\sigma 2$ = 0.07, what is the optimal value of q? Here $\mu 2$ is the expected value of the random variable R_2 and $\sigma 2$ is the standard deviation of R_2.

e. Show that with the negative exponential utility function and normally distributed returns, the optimal q is the same as that obtained in part d. What are the expected returns in this case?

f. Change the standard deviation of returns to 0.15 and solve part e again.

g. Change the standard deviation back to 0.07 and change a to 50 and solve part e again.

h. Explain the answers in parts d, e, f, and g.

i. Now repeat part e assuming a Beta distribution having the same mean and standard deviation as in part d. The boudns of the beta dsitribution are zero and 1. You can use the fact that

$$\begin{aligned} Mean\ [Beta\ Distribution\,] &= \frac{\alpha}{\alpha + \beta} \\ Variance\ [Beta\ Distribution\,] &= \frac{\alpha\,\beta}{(\alpha + \beta)^2\,(\alpha + \beta + 1)} \end{aligned}$$

j. Change the standard deviation of returns to 0.15 and solve part i again.

k. Now repeat part e assuming a Gamma distribution having the same mean and standard deviation as in part d. The bounds of a gamma distribution are zero and infinity. You can use the fact that

$$Mean \ [Gamma \ Distribution \ ] \ = \alpha \, \beta$$

$$Variance \ [Gamma \ Distribution \ ] \ = \alpha \, \beta^2$$

l. Discuss the differences in returns and portfolio proportions from parts d-k.