Economics 571  
Problem Set #2 Solutions  

(1a)  
\[ \Pr(X \leq 7) = \Pr(\frac{X - 3}{4} \leq \frac{7 - 3}{4}) = \Pr(Z \leq -1) = .841 \]  

In the above \( Z \sim N(0,1) \), and the last entry is read from the Normal tables (815 and 816 of your book).  

(1b)  
\[ \Pr(X > 5) = \Pr(\frac{X - 3}{4} > \frac{5 - 3}{4}) = \Pr(Z > 1/2) = 1 - \Pr(Z \leq 1/2) = 1 - .6915 = .3085 \]  

(1c)  
\[ \Pr(|X| \leq 3) = \Pr(-3 \leq X \leq 3) = \Pr(X \leq 3) - \Pr(X \leq -3) \]  
\[ = \Pr(\frac{|X - 3|}{4} \leq 0) - \Pr(\frac{|X - 3|}{4} \leq -6/4) \]  
\[ = 1/2 - \Pr(Z \leq -3/2) = 1/2 - .067 \]  
\[ = .433 \]  

(2) Since \( Y \) is also normal, it remains to calculate its mean and variance. Note:  
\[ E(Y) = E[3 - (1/4)X] = 3 - (1/4)E(X) = 3 - 3/4 = 9/4 = 2.25 \]  
and  
\[ \text{Var}(Y) = \text{Var}(3 - (1/4)X) = \text{Var}(\frac{1}{4}X) = (1/16)\text{Var}(X) = 1. \]  

Thus,  
\[ Y \sim N(9/4,1). \]  

So,  
\[ \Pr(3.25 < Y < 4.25) = \Pr(Y < 4.25) - \Pr(Y < 3.25) \]
\[
\begin{align*}
&= \Pr(Y - 2.25 < 4.25 - 2.25) - \Pr(Y - 2.25 < 3.25 - 2.25) \\
&= \Pr(Z < 2) - \Pr(Z < 1) \\
&= .9772 - .8413 \\
&= .1359.
\end{align*}
\]

(b)

\[
\begin{align*}
\Pr(Y > X) &= \Pr(3 - (1/4)X > X) \\
&= \Pr(3 > (5/4)X) \\
&= \Pr(X < 12/5) \\
&= \Pr(Z < [(12/5) - 3]/4) \\
&= \Pr(Z < -.15) \\
&= .4404.
\end{align*}
\]

(3a)

\[
E(\hat{\mu}) = E[cy_1 + dy_2] = cE(y_1) + dE(y_2) = (c + d)\mu.
\]

For this to be an unbiased estimator of \(\mu\),

\[
c + d = 1,
\]

or equivalently,

\[
d = (1 - c).
\]

(3b) Under the restriction in (a),

\[
\text{Var}(cy_1 + [1 - c]y_2) = c^2\text{Var}(y_1) + (1 - c)^2\text{Var}(y_2) = \sigma^2[c^2 + (1 - c)^2].
\]

(3c) Differentiating the above expression with respect to \(c\) and setting this equal to zero gives:

\[
2c^* - 2(1 - c^*) = 0,
\]

or equivalently,

\[
4c^* = 2,
\]

implying that \(c^* = 1/2\). Thus, our formula reduces to the sample average of these two observations. The result in (3c) suggests that the sample average is efficient among the class of linear, unbiased estimators of \(\mu\).
(4) This problem is not as tricky as it seems, once a reasonable counterexample has been found. For example, suppose we have a sample of observations $y_1, y_2, \ldots, y_n$ from a population with mean $\mu$ with $E(y_i) = \mu$ and $\text{Var}(y_i) = \sigma^2$ for all $i$. Consider

$$\hat{\mu} = y_1.$$ 

Clearly, this is an unbiased estimator of $\mu$ since $E(\hat{\mu}) = \mu$. However, this is not consistent: $\text{Var}(\hat{\mu}) = \sigma^2$. The sampling distribution of the estimator is just the population density, which clearly does not collapse around anything. As the sample size increases, we continue to use only the information in one observation. So, we have an unbiased, but inconsistent estimator.

Another such estimator is of the form:

$$\hat{\mu} = \begin{cases} 
\mu - 10 & \text{with probability } 1/2 \\
\mu + 10 & \text{with probability } 1/2
\end{cases}$$

This, again is unbiased, but clearly not consistent.

(5) Intuitively, we make the assumption that

$$\Pr(Y = x/2|X = x) = 1/2, \quad \Pr(Y = 2x|X = x) = 1/2$$

regardless of the value of $x$. However, this can not be the case. When $X = m$, for example,

$$\Pr(Y = 2X|X = m) = 1$$

while when $X = 2m$,

$$\Pr(Y = X/2|X = 2m) = 1.$$ 

Working with these conditional distributions, one can show that the *ex ante* expected gain from switching is $3m/2$, which is the same as the *ex ante* expected gain from not switching!

(5b) First note that $m$ here is the “parameter” to be learned about, and the data we have is $X = x$. So, ultimately, we want to update our prior beliefs about $m$ in light of what we see in $x$. Finally, since $x$ is observed, it must be the case that $M$ either equals $x$ or $x/2$, depending upon whether or not we opened the larger or smaller envelope.

Note by simple probability manipulations:

$$\Pr(M = x|X = x) = \frac{\Pr(X = x|M = x)\Pr(M = x)}{\Pr(X = x)}$$

$$= \frac{\Pr(X = x|M = x)\Pr(M = x)}{\Pr(X = x|M = x)\Pr(M = x) + \Pr(X = x|M = x/2)\Pr(M = x/2)}$$

$$= \frac{p(x)}{p(x) + p(x/2)}$$
Similarly,
\[ \Pr(M = x/2|X = x) = \frac{p(x/2)}{p(x/2) + p(x)} \]

Now, consider the option of switching. The second probability above is the (posterior) probability that I have the “big” prize, while the first is the posterior probability that I have the small prize. So, by switching, I would essentially reverse these probabilities. Thus, my expected return from switching is:
\[ \frac{p(x)}{p(x) + p(x/2)} \cdot 2x + \frac{p(x/2)}{p(x/2) + p(x)} \cdot \frac{x}{2} \]

I would choose to switch if this exceeded \( x \). Performing the needed algebra, one can then show that a switch is optimal provided:
\[ 2p(x) > p(x/2) \]

For the case of an exponential density, this implies that
\[ 2\lambda \exp(-\lambda x) > \lambda \exp(-\lambda x/2) \]

Rearranging gives
\[ x < 2 \log(2)/\lambda \]

Thus, if \( x \) is sufficiently small, then you will decide to switch. In addition, as \( \lambda \) increases, the right-hand side decreases, thus lowering the switching threshold. Note that the prior mean is \( \lambda^{-1} \) so that a large \( \lambda \) indicates that you are believe the prize is likely to be small. In such a case, you would lower your threshold in light of these beliefs.