(1a) \[ \Pr(X \leq 7) = \Pr\left( \frac{X - 3}{4} \leq \frac{7 - 3}{4} \right) = \Pr(Z \leq -1) = .841 \]

In the above \( Z \sim N(0, 1) \), and the last entry is read from the Normal tables (815 and 816 of your book).

(1b) \[ \Pr(X > 5) = \Pr\left( \frac{X - 3}{4} > \frac{5 - 3}{4} \right) = \Pr(Z > 1/2) = 1 - \Pr(Z \leq 1/2) = 1 - .6915 = .3085 \]

(1c) \[ \Pr(|X| \leq 3) = \Pr(-3 \leq X \leq 3) = \Pr(X \leq 3) - \Pr(X \leq -3) = \Pr(\frac{X - 3}{4} \leq 0) - \Pr(\frac{X - 3}{4} \leq -6/4) = 1/2 - \Pr(Z \leq -3/2) = 1/2 - .067 = .433 \]

(2) Since \( Y \) is also normal, it remains to calculate its mean and variance. Note:

\[ E(Y) = E[3 - (1/4)X] = 3 - (1/4)E(X) = 3 - 3/4 = 9/4 = 2.25 \]

and

\[ \text{Var}(Y) = \text{Var}(3 - (1/4)X) = \text{Var}((1/4)X) = (1/16)\text{Var}(X) = 1. \]

Thus, \( Y \sim N(9/4, 1) \).

So,

\[ \Pr(3.25 < Y < 4.25) = \Pr(Y < 4.25) - \Pr(Y < 3.25) \]
\[
\begin{align*}
= \Pr(Y - 2.25 < 4.25 - 2.25) - \Pr(Y - 2.25 < 3.25 - 2.25) \\
= \Pr(Z < 2) - \Pr(Z < 1) \\
= .9772 - .8413 \\
= .1359.
\end{align*}
\]

(b)
\[
\begin{align*}
\Pr(Y > X) &= \Pr(3 - (1/4)X > X) \\
&= \Pr(3 > (5/4)X) \\
&= \Pr(X < 12/5) \\
&= \Pr(Z < [(12/5) - 3]/4) \\
&= \Pr(Z < -.15) \\
&= .4404
\end{align*}
\]

(3 a) We have shown in class that MSE = Bias\(^2\) + Variance. So, let us first calculate each of these pieces.

As for the bias:
\[
E(\hat{\theta}) = \frac{1}{n} \left[ \sum_{i=1}^{n_1} E(x_{1i}) + \sum_{i=n_1+1}^{n} E(x_{2i}) \right] = \frac{1}{n} [n_1(\theta + c) + (n - n_1)\theta].
\]

Re-arranging terms we can write
\[
E(\hat{\theta}) = \theta + \frac{cn_1}{n}.
\]

Thus, viewed according to only the (absolute) bias, we would clearly want to set \(n_1 = 0\) and thus use only observations from group 2.

As for the variance,
\[
\text{Var}(\hat{\theta}) = \frac{1}{n^2} \text{Var} \left[ \sum_{i=1}^{n_1} \left( x_{1i} \right) + \sum_{i=n_1+1}^{n} x_{2i} \right].
\]

Because each of the \(x_i\) are independent of one another, the variance of this sum is simply the sum of the variances so that
\[
\text{Var}(\hat{\theta}) = \frac{1}{n^2} \left[ \sum_{i=1}^{n_1} \text{Var}(x_{1i}) + \sum_{i=n_1+1}^{n} \text{Var}(x_{2i}) \right].
\]

Thus
\[
\text{Var}(\hat{\theta}) = \frac{1}{n^2} [n_1\sigma_1^2 + (n - n_1)\sigma_2^2]
\]
Thus, viewed according to only the variance, we would want to make $n_1$ as large as possible (i.e., $n_1 = n$) to minimize this expression.

The MSE criterion balances both of these factors. Using the above results, we obtain

$$MSE(\hat{\theta}) = \frac{c^2 n_1^2}{n^2} + \frac{1}{n^2}[n\sigma_1^2 + (n - n_1)d].$$

Differentiating this result with respect to $n_1$ and setting the resulting expression equal to zero, we obtain

$$n_1^* = \frac{d}{2c^2}.$$

This result shows that the optimal $n_1$ clearly increases with the variability gap $d$ and decreases with the extent of the bias $c$.

(4) This problem is not as tricky as it seems, once a reasonable counterexample has been found. For example, suppose that

$$\hat{\theta} = \theta + \frac{1}{n}.$$ 

It is clear that this is a consistent estimator since, for any $\epsilon > 0$, we can always find an $n$ sufficiently large such that this estimator places all of its mass within $\epsilon$ of $\theta$.

It is biased, however, since $E(\hat{\theta}) = \theta + \frac{1}{n}$ $\neq \theta$. Note that bias refers to average performance for a given $n$ while consistency deals with the sampling distribution as $n$ gets arbitrarily large.