Normal distribution

Definition: Continuous rv with bell-shaped probability density function:

\[ f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \]

where \( \pi = 3.1415... \)
\( \exp(a) = e^a, \) with \( e = 2.7182... \)
\( \mu, \sigma^2 = \) parameters

Abbreviation: \( X \sim N(\mu, \sigma^2) \)

Properties:

(1) \( E(X) = \mu, \) Var(\(X\)) = \( \sigma^2, \) SD(\(X\)) = \( \sigma. \)

(2) \( f(x; \mu, \sigma^2) \) is bell-shaped curve, symmetric about \( \mu \) (median).
(3) \( f(x; \mu, \sigma^2) \) has peak at \( \mu \) (mode).
(4) \( f(x; \mu, \sigma^2) \) is flatter for larger values of \( \sigma. \)
(5) \( f(x; \mu, \sigma^2) > 0 \) for all \( x. \) However \( f(x; \mu, \sigma^2) \) is very small for \( |x-\mu| > 2\sigma. \)
More properties of normal distribution

(6) If \( X_1 \sim N(\mu, \sigma^2) \), then \( X_2 = aX_1 + b \) is also normally distributed.

Now by properties of mean and variance we have:
\[
E(X_2) = aE(X_1) + b = a\mu + b \\
Var(X_2) = a^2 Var(X_1) = a^2 \sigma^2
\]

So \( X_2 \sim N( a\mu+b, a^2\sigma^2 ) \)

From this property, it follows that if \( X \sim N(\mu, \sigma^2) \),
then \( Z = (X-\mu)/\sigma \sim N(0,1) \).

The special normal distribution \( N(0,1) \) is called the standard normal.

The operation of subtracting the mean and dividing by the SD is called standardizing. Using this trick we can find probabilities for any normal random variable using only the table for the standard normal.
Standardizing a normal rv

Example 1: Suppose $X \sim N(24,9)$ and we wish to find $\text{Prob}(X < 15)$. Now this equals:

$$= \text{Prob} \left( \frac{X - 24}{3} < \frac{15 - 24}{3} \right)$$

$$= \text{Prob} (Z < -3)$$

where $Z \sim N(0,1)$.

From standard normal table, $\text{Prob}(Z > 3) = 0.0013$.

By symmetry, $\text{Prob}(Z < -3) = 0.0013$ also.
Standardizing a normal rv

Example 2: Suppose again \( X \sim N(24,9) \) but now we wish to find \( \text{Prob}( 24 < X < 26 ) \). Now this equals:

\[
= \text{Prob} \left( \frac{24 - 24}{3} < \frac{X - 24}{3} < \frac{26 - 24}{3} \right)
\]

\[
= \text{Prob} \ (0 < Z < 0.67)
\]

where \( Z \sim N(0,1) \).

From standard normal table, \( \text{Prob}(Z>0.67) = 0.2514 \).

Since \( \text{Prob}(Z>0) = 0.5 \), it follows that:

\[
\text{Prob}(0 < Z < 0.67) = 0.5 - 0.2514 = 0.2486.
\]
More properties of normal distribution

(7) If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then any linear combination of the two, say $X_3 = aX_1 + bX_2$, is also normally distributed.

Now by properties of mean and variance we have:

$E(aX_1 + bX_2) = a\mu_1 + b\mu_2$

$\text{Var}(aX_1 + bX_2) = a^2\sigma_1^2 + b^2\sigma_2^2$

So $X_3 \sim N( a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2 )$
More properties of normal distribution

(8) Central limit theorem: Suppose $X_1, \ldots, X_n$ are independent identically-distributed rvs (not necessarily normal), each with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. Then the sample mean:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{A} N\left(\mu, \frac{\sigma^2}{n}\right)$$

The "A" over the "\xrightarrow{}" indicates "asymptotically," or approximately for large $n$.

Alternatively, we can express this result after standardizing $\bar{X}$:

$$Z = \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \xrightarrow{A} N\left(0, 1\right)$$
**Chi-square ($\chi^2$) distribution**

Definition: Suppose $Z_1, \ldots, Z_n$ are independent and each $\sim N(0, 1)$. Then:

$$Y = \sum_{i=1}^{n} Z_i^2$$

distributed as "chi-square with $n$ degrees of freedom (DOF)."

Abbreviation: $Y \sim \chi^2(n)$

Properties:

(1) $E(Y) = n$, $\text{Var}(Y) = 2n$.  

(2) $Y \geq 0$, so distribution is skewed to right. However, becomes more symmetric for large $n$.

(3) If $Y_1 \sim \chi^2(n_1)$ and $Y_2 \sim \chi^2(n_2)$ are independent, then $Y = Y_1 + Y_2 \sim \chi^2(n_1+n_2)$. 
t distribution

Definition: Suppose \( Z \sim N(0,1) \) and \( Y \sim \chi^2(n) \) are independent. Then:

\[
W = \frac{Z}{\sqrt{Y/n}}
\]

distributed as "t with n DOF."

Abbreviation: \( W \sim t(n) \)

Properties:

(1) Density function is bell-shaped curve, symmetric around zero (median).

(2) Density has peak at zero (mode).

(3) Density > 0 for all \( x \).

(4) \( E(W) = 0 \) for \( n \geq 2 \), but \( \text{Var}(W) = n/(n-2) > 1 \).

(5) As \( n \to \infty \), \( t(n) \to N(0,1) \).
**F distribution**

**Definition:** Suppose $Y_1 \sim \chi^2(n_1)$ and $Y_2 \sim \chi^2(n_2)$ are independent. Then:

$$V = \frac{Y_1/n_1}{Y_2/n_2}$$

distributed as "F with $n_1$ DOF in numerator and $n_2$ DOF in denominator."

**Abbreviation:** $V \sim F(n_1,n_2)$

**Properties:**

1. $V \geq 0$, so distribution is skewed to right. However, becomes more symmetric for large $n_1$.

2. If $W \sim t(n)$, then $W^2 \sim F(1,n)$.

3. $E(V) = n_2/(n_2-2)$. Thus $E(V) \rightarrow 1$ as $n_2 \rightarrow \infty$. 