Exam #1

Do all three problems. Weights: #1 - 25%, #2 - 35%, #3 - 40%. Closed book, closed notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in the wording.

a. Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable. $F(\cdot)$ is concave if and only if

$$F(u) < F(v) + \frac{\partial F}{\partial x}(v)(u - v) \quad \text{for all } u, v \in \mathbb{R}^n, u \neq v.$$

b. For $F : \mathbb{R} \rightarrow \mathbb{R}$, if $F(\cdot)$ is either strictly increasing or strictly decreasing, then $F(\cdot)$ is strictly quasi-concave.

c. $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable and $x^* \in \mathbb{R}^n$. If $\frac{\partial F}{\partial x}(x^*) = 0$ and $\frac{\partial^2 F}{\partial x^2}(x^*)$ is negative definite then $x^*$ is a strict local maximum of $F(\cdot)$.

d. In an equality-constrained optimization problem with 3 choice variables ($n = 3$) and 2 constraints ($m = 2$), the bordered Hessian is a $5 \times 5$ matrix and the second order necessary condition is a single sign restriction on the determinant of the entire bordered Hessian.

e. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable and let $x^* \in \mathbb{R}^n$ be a point such that $\frac{\partial g}{\partial x}(x^*) \neq 0$. Then $x^*$ is a local solution to

$$\max_{w.r.t. x} F(x) \quad \text{such that} \quad g(x) = b$$

if and only if there exists $\lambda^* \in \mathbb{R}$ such that $(x^*, \lambda^*)$ is a stationary point of

$$L(x, \lambda) \equiv F(x) + \lambda[b - g(x)].$$
2. Consider the problem:

\[ \min_{\text{w.r.t. } x} \ a \cdot x \quad \text{such that} \quad x \in U, \]

where \( x \) is an \( n \times 1 \) vector of choice variables, \( a \) is a \( 1 \times n \) vector of parameters, and \( U \subset \mathbb{R}^n \). Suppose that the problem has a global solution, \( x^*(a) \), for all \( a \in \mathbb{R}^n \), and define the value function \( F^*(a) \equiv a \cdot x^*(a) \).

(Some facts that you don't need to prove: A sufficient condition for the problem to have a global solution for all \( a \in \mathbb{R}^n \) is that \( U \) is closed and bounded. While the optimal value of \( x \), \( x^*(a) \), need not be unique for a given \( a \), the value function is uniquely defined. That is, if, for a given \( a \), there are multiple optimal \( x^* \)'s, they all yield the same value of the objective function.)

Prove that \( F^* : \mathbb{R}^n \to \mathbb{R} \) is concave.

3. Consider the following equality-constrained maximization problem:

\[ \max_{\text{w.r.t. } x_1, x_2, x_3} \ f(x_1, x_2, x_3) \quad \text{such that} \quad g(x_1, x_2, x_3) = b \quad (i.) \]

where \( f : \mathbb{R}^3 \to \mathbb{R} \) and \( g : \mathbb{R}^3 \to \mathbb{R} \) are differentiable functions and \( b \) is a constant. Also consider the unconstrained optimization problem:

\[ \max_{\text{w.r.t. } x_1, x_2} \ F(x_1, x_2) \quad (ii.) \]

where \( F : \mathbb{R}^2 \to \mathbb{R} \) is differentiable.

a. Write down the Lagrangian and the first-order conditions for problem (i.).

b. Write down the first-order conditions for problem (ii.).

Now suppose that \( g(x_1, x_2, x_3) = x_3 - h(x_1, x_2) \) where \( h : \mathbb{R}^2 \to \mathbb{R} \) is differentiable, and that \( F(x_1, x_2) = f(x_1, x_2, h(x_1, x_2) + b) \). For this case, show that

c. if \( (x_1^*, x_2^*, x_3^*) \) satisfies the first-order conditions for problem (i.), then \( (x_1^*, x_2^*) \) satisfies the first-order conditions for problem (ii.).

d. if \( (x_1^*, x_2^*) \) satisfies the first-order conditions for problem (ii.), then \( (x_1^*, x_2^*, h(x_1^*, x_2^*) + b) \) satisfies the first-order conditions for problem (i.).