Exam #1 Solution Outline

1. a. False. Change “negative definite” to “negative semi-definite.”

b. False. There are several ways to correct it with small changes. One way:

“If \( F(\cdot) \) is strictly concave then it is both strictly quasi-concave and quasi-concave.”

c. True.  
d. True.

e. False. Change “. . . is negative semi-definite” to “. . . has non-negative determinant.”

2. Proof: For any \( x_1, x_2 \in \mathbb{R}^n \) and \( h \in [0, 1] \),

\[
F(hx_1 + (1-h)x_2) = [hF(x_1) + (1-h)F(x_2)] = \\
(hx_1 + (1-h)x_2)'A(hx_1 + (1-h)x_2) - hx_1'Ax_1 - (1-h)x_2'Ax_2 \\
= h^2x_1'Ax_1 + 2h(1-h)x_1'Ax_2 + (1-h)^2x_2'Ax_2 - hx_1'Ax_1 - (1-h)x_2'Ax_2 \\
= h(h-1)[x_1'Ax_1 - 2x_1'Ax_2 + x_2'Ax_2] \\
= h(h-1)(x_1 - x_2)'A(x_1 - x_2) \equiv \Delta
\]

\( A \) negative semi-definite \( \Rightarrow \Delta \geq 0 \) for all \( x_1, x_2 \in \mathbb{R}^n \) and \( h \in [0, 1] \). That is, \( F(\cdot) \) is concave.

\( F(\cdot) \) concave \( \Rightarrow \Delta \geq 0 \) for \( h = 0.5, x_2 = 0 \), and for all \( x_1 \in \mathbb{R}^n \). This implies \( x_1'Ax_1 \leq 0 \) for all \( x_1 \in \mathbb{R}^n \), that is, \( A \) is negative semi-definite.

3. Let \( x_1, x_2, \ldots, x_n \) denote the quantities of the inputs. Let \( p, w_1, w_2, \ldots, w_n \) denote the prices of output, and the \( n \) inputs. Let \( F(\cdot) \) denote the production function. The profit maximization problem is:

\[
\max_{w.r.t. x_1, x_2, \ldots, x_n} \pi(x; p, w) \text{ where } \pi(x; p, w) = pF(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i
\]
The assumption that the problem has a regular solution for each price vector means that factor demands exist as differentiable functions:

\[ x_i^*(p, w_1, w_2, \ldots, w_n) \quad \text{for} \quad i = 1, 2, \ldots, n. \]

Define the value function: \( \pi^*(p, w) \equiv \pi(x^*(p, w); p, w) \). By the envelope theorem:

\[
\frac{\partial \pi^*}{\partial p}(p, w) = \frac{\partial \pi}{\partial p}(x^*(p, w); p, w) = F(x^*(p, w)) \equiv q^*(p, w),
\]

where \( q^*(p, w) \) denotes the firm’s supply function, and

\[
\frac{\partial \pi^*}{\partial w_i}(p, w) = \frac{\partial \pi}{\partial w_i}(x^*(p, w); p, w) = -x_i^*(p, w).
\]

Taking second derivatives and using Young’s theorem:

\[
\frac{\partial q^*}{\partial w_i}(p, w) = \frac{\partial^2 \pi^*}{\partial w_i \partial p}(p, w) = \frac{\partial^2 \pi^*}{\partial p \partial w_i}(p, w) = -\frac{\partial x_i^*}{\partial p}(p, w).
\]

So \( \frac{\partial q^*}{\partial w_i}(p, w) \) and \( \frac{\partial x_i^*}{\partial p}(p, w) \) are of opposite sign. Thus, an increase in output price raises the demand for a given input if and only if an increase in the price of the input reduces optimal output.