Econ 600
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Fall 2008

Exam #1 Solution Outline

1. a. False. Change "negative definite" to "negative semi-definite."

b. False. There are several ways to correct it with small changes. One way:

"If $F(\cdot)$ is strictly concave then it is both strictly quasi-concave and quasi-concave."

c. True. d. True.

e. False. Change " . . . is negative semi-definite" to " . . . has non-negative determinant."

2. Proof: For any $x_1, x_2 \in \Re^n$ and $h \in [0, 1]$,

$$F(hx_1 + (1-h)x_2) - [hF(x_1) + (1-h)F(x_2)] =$$
$$(hx_1 + (1-h)x_2)' A(hx_1 + (1-h)x_2) - hx_1' Ax_1 - (1-h)x_2' Ax_2$$
$$= h^2 x_1' Ax_1 + 2h(1-h)x_1' Ax_2 + (1-h)^2 x_2' Ax_2 - hx_1' Ax_1 - (1-h)x_2' Ax_2$$
$$= h(h-1)\left[x_1' Ax_1 - 2x_1' Ax_2 + x_2' Ax_2 \right]$$
$$= h(h-1)(x_1 - x_2)' A(x_1 - x_2) \equiv \Delta$$

A negative semi-definite $\Rightarrow \Delta \geq 0$ for all $x_1, x_2 \in \Re^n$ and $h \in [0, 1]$. That is, $F(\cdot)$ is concave.

$F(\cdot)$ concave $\Rightarrow \Delta \geq 0$ for $h = 0.5$, $x_2 = 0$, and for all $x_1 \in \Re^n$. This implies $x_1' Ax_1 \leq 0$ for all $x_1 \in \Re^n$, that is, A is negative semi-definite.

3. Let $x_1, x_2, \ldots, x_n$ denote the quantities of the inputs. Let $p; w_1, w_2, \ldots, w_n$ denote the prices of output, and the n inputs. Let $F(\cdot)$ denote the production function. The profit maximization problem is:

$$\max_{w.r.t.\ x_1, x_2, \ldots x_n} \pi(x; p, w) \quad \text{where} \quad \pi(x; p, w) \equiv pF(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i$$

The assumption that the problem has a regular solution for each price vector means that factor demands exist as differentiable functions:

$$x_i^*(p, w_1, w_2, \ldots, w_n) \text{ for } i = 1, 2, \ldots, n.$$

Define the value function: $\pi^*(p,w) \equiv \pi\left(x^*(p,w); p, w\right)$. By the envelope theorem:

$$\frac{\partial \pi^*}{\partial p}(p,w) = \frac{\partial \pi}{\partial p}\left(x^*(p,w); p, w\right) = F\left(x^*(p,w)\right) \equiv q^*(p,w),$$

where $q^*(p,w)$ denotes the firm's supply function, and

$$\frac{\partial \pi^*}{\partial w_i}(p,w) = \frac{\partial \pi}{\partial w_i}\left(x^*(p,w); p, w\right) = -x_i^*(p,w).$$

Taking second derivatives and using Young's theorem:

$$\frac{\partial q^*}{\partial w_i}(p,w) = \frac{\partial^2 \pi^*}{\partial w_i \partial p}(p,w) = \frac{\partial^2 \pi^*}{\partial p \partial w_i}(p,w) = -\frac{\partial x_i^*}{\partial p}(p,w).$$

So $\frac{\partial q^*}{\partial w_i}(p,w)$ and $\frac{\partial x_i^*}{\partial p}(p,w)$ are of opposite sign. Thus, an increase in output price raises the demand for a given input if and only if an increase in the price of the input reduces optimal output.