Econ 600
Schroeter
Fall 2011

Exam #1

Do all three problems. Weights: #1 - 30%, #2 - 35%, #3 - 35%.
Closed book, closed notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer **True** or **False** for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

a. Let U be a convex subset of $\Re^n$. If $F: U \to \Re$ is differentiable then

$$F(\cdot) \text{ is concave if and only if } F(u) \le F(v) + \frac{\partial F}{\partial x}(v)(u-v) \text{ for all } u, v \in U.$$

b. Let U be a convex subset of $\Re^n$. If $F: U \to \Re$ is differentiable then

$$F(\cdot) \text{ is strictly concave if and only if } \frac{\partial^2 F}{\partial x^2}(v) \text{ is negative definite for all } v \in U.$$

c. Let $F: \Re^n \to \Re$ be differentiable. If $x^* \in \Re^n$ is such that $\frac{\partial F}{\partial x}(x^*) = 0$ and $\frac{\partial^2 F}{\partial x^2}(x^*)$ is positive semi-definite then x^* is a local minimum of $F(\cdot)$.

d. Let $F: \Re^2 \to \Re$ and $g: \Re^2 \to \Re$ be differentiable and let $b \in \Re$. Let $x^* \in \Re^2$ be a point at which $\frac{\partial g}{\partial x}(x^*) \ne 0$ and a local solution to

$$\max_{w.r.t.\, x} F(x) \text{ such that } g(x) = b.$$

Then there exists $\lambda^* \in \Re$ such that $\frac{\partial F}{\partial x}(x^*) = \lambda^* \frac{\partial g}{\partial x}(x^*)$.

e. $F: \Re \; x \; \Re \to \Re$ is differentiable. For each value of the scalar parameter *a*, $\min_{w.r.t.\, x} F(x; a)$ has a regular solution at $x^*(a)$. Define: $F^*(a) \equiv F(x^*(a); a)$. Then, for each value of the parameter a_0:

$$\frac{\partial^2}{\partial a^2} F(x^*(a_0); a_0) \ge \frac{d^2}{da^2} F^*(a_0).$$

2. First recall the following definition of a quasi-convex function:

Let U be a convex subset of $\Re^n$. $F : U \to \Re$ is <u>quasi-convex</u> if, for all $u, v \in U$, and for all $h \in [0, 1]$,

$$F(hu + (1-h)v) \le \max\{F(u), F(v)\}.$$

Now, for U a convex subset of $\Re^n$, let $F : \Re^m \, x \, U \to \Re$ be such that $F(x; a)$ is quasi-convex in a for every $x \in \Re^m$. Consider the problem $\max_{w.r.t.\, x} F(x; a)$ and assume that it has a global solution $x^*(a)$ for each $a \in U$. Define the value function in the usual way: $F^*(a) \equiv F(x^*(a); a)$. (Note that $F^*(a)$ is well-defined even if $x^*(a)$ is not unique.)

Prove that $F^* : U \to \Re$ is quasi-convex.

3. Consider the problem:

$$\min_{w.r.t.\, x_1,\, x_2} a_1 x_1 + a_2 x_2 \quad \text{such that} \quad g(x_1, x_2) = c,$$

where c is a constant and a_1 and a_2 are positive constants. The constraint function, $g : \Re^2 \to \Re$ is differentiable with first- and second-order partial derivatives (denoted using subscripts) that satisfy the following sign restrictions throughout $\Re^2$:

$$g_1, g_2 > 0; \quad g_{11}, g_{22} < 0; \quad g_{12} > 0; \quad \text{and} \quad g_{11} g_{22} - g_{12}^2 > 0.$$

a. Write down the Lagrangian and the first order necessary conditions for this problem. The assumptions made above insure that the second-order sufficient conditions for a strict local minimum are satisfied at any point that satisfies the first-order necessary conditions (although you don't have to show this).

b. Use $x_1^*(a_1, a_2)$, $x_2^*(a_1, a_2)$, and $\lambda^*(a_1, a_2)$ to denote the optimal values of x_1 and x_2, and the solution value for the Lagrange multiplier, as functions of a_1 and a_2. Show that

$$\frac{\partial \lambda^*}{\partial a_1} > 0 \quad \text{and} \quad \frac{\partial \lambda^*}{\partial a_2} > 0.$$