Exam #1

Do all three problems. Weights: #1 - 30%, #2 - 35%, #3 - 35%.
Closed book, closed notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

   a. Let $U$ be a convex subset of $\mathbb{R}^n$. If $F : U \to \mathbb{R}$ is differentiable then

   $F(\cdot)$ is strictly concave if and only if $F(u) < F(v) + \frac{\partial F}{\partial x}(v-u)$ for all $u, v \in U, u \neq v$.

   b. If $F : \mathbb{R}^n \to \mathbb{R}$ is differentiable then $x^*$ is a local solution to $\max_{x \in U} F(x)$ if and only if

   \[
   \frac{\partial F}{\partial x}(x^*) = 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(x^*) \text{ is negative semi-definite.}
   \]

   c. If $F : \mathbb{R} \to \mathbb{R}$ is differentiable and strictly quasi-concave, and $x^* \in \mathbb{R}$ is such that $F''(x^*) = 0$, then $x^*$ cannot be a local minimum.

   d. If $F : \mathbb{R}^n \to \mathbb{R}$ is strictly quasi-concave and concave, then it is also strictly concave.

   e. $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Suppose that the problem:

   \[
   \max_{x_1, x_2} F(x_1, x_2) \quad \text{subject to} \quad g(x_1, x_2) = b
   \]

   has a regular solution, $x^*(b) = (x_1^*(b), x_2^*(b))$, for all $b \in \mathbb{R}$. Define: $F^*(b) = F(x^*(b))$.

   Suppose that, at a particular value of $b = b_0$, \( \frac{\partial F}{\partial x_1}(x^*(b_0)) \) and \( \frac{\partial g}{\partial x_1}(x^*(b_0)) \) are both negative. Then $\frac{dF^*}{db}(b_0) < 0$. 
2. $U$ is a convex subset of $\mathbb{R}^n$, and $F: \mathbb{R}^n \times U \to \mathbb{R}$ is such that $F(x; a)$ is quasi-concave in $a$ for every $x \in \mathbb{R}^m$. Consider the problem $\min_{x, \text{w.r.t. } x} F(x; a)$ and assume that it has a global solution $x^*(a)$ for each $a \in U$. Define the value function in the usual way: $F^*(a) \equiv F(x^*(a); a)$. (Note that $F^*(a)$ is well-defined even if $x^*(a)$ is not unique.)

Prove that $F^*: U \to \mathbb{R}$ is quasi-concave.

3. $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R}^2 \to \mathbb{R}$ are differentiable. Suppose that the problem:

$$\max_{x, \text{w.r.t. } x_1, x_2; \text{ given } \alpha} f(x_1, x_2; \alpha) \text{ subject to } g(x_1, x_2) = 0$$

has a regular solution, $x^*(\alpha) = (x_1^*(\alpha), x_2^*(\alpha))$, for all $\alpha \in \mathbb{R}$. At a particular value of $\alpha = \alpha_0$:

$$\frac{\partial^2 f}{\partial x_2 \partial \alpha}(x^*(\alpha_0); \alpha_0) = 0 \text{ and } \frac{\partial g}{\partial x_1}(x^*(\alpha_0)) \cdot \frac{\partial g}{\partial x_2}(x^*(\alpha_0)) > 0.$$

a. Show that $\frac{dx_1^*}{d\alpha}(\alpha_0)$ has the same algebraic sign as $\frac{\partial^2 f}{\partial x_1 \partial \alpha}(x^*(\alpha_0); \alpha_0)$.

b. Show that $\frac{dx_2^*}{d\alpha}(\alpha_0)$ has the same algebraic sign as $-\frac{\partial^2 f}{\partial x_1 \partial \alpha}(x^*(\alpha_0); \alpha_0)$. 