Econ 600
Schroeter
Fall 2011

Exam #1 Solution Outline

1. *a*. **True**. *b*. **False.** Change "if and only if" to "if."

c. **False**. Change "positive semi-definite" to "positive definite." (And then could also strengthen the claim by changing "local minimum" to "strict local minimum.")

d. **True.** *e*. **True.**

2. Claim: $F^*: U \to \Re$ is quasi-convex.

Given arbitrary $u, v \in U$ and $h \in [0, 1]$, need to show:

$$F^*(hu + (1-h)v) \le \max\{F^*(u), F^*(v)\}.$$

$$F^*(hu + (1-h)v) = F(x^*(hu + (1-h)v); hu + (1-h)v) \le$$
$$\max\{F(x^*(hu + (1-h)v); u), F(x^*(hu + (1-h)v); v)\}$$

(by the quasi-convexity of $F(x; a)$ in a for every $x \in \Re^m$)

$$F(x^*(hu + (1-h)v); u) \le F(x^*(u); u) \quad \text{(because } x^*(u) = \underset{w.r.t.\,x}{\arg\max}\ F(x; u)\text{)}$$

Likewise: $F(x^*(hu + (1-h)v); v) \le F(x^*(v); v)$

So

$$\begin{aligned} F^*(hu + (1-h)v) &\le \max\{F(x^*(hu + (1-h)v); u), F(x^*(hu + (1-h)v); v)\} \\ &\le \max\{F(x^*(u); u), F(x^*(v); v)\} \\ &= \max\{F^*(u), F^*(v)\} \qquad \textit{Q.E.D.} \end{aligned}$$

3. *a*. $L(x_1, x_2; \lambda) = a_1 x_1 + a_2 x_2 + \lambda(c - g(x_1, x_2))$

FONC:

$$\frac{\partial L}{\partial \lambda}(x_1^*, x_2^*; \lambda^*) = c - g(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial x_1}(x_1^*, x_2^*; \lambda^*) = a_1 - \lambda^* g_1(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial x_2}(x_1^*, x_2^*; \lambda^*) = a_2 - \lambda^* g_2(x_1^*, x_2^*) = 0$$

Note: The fact that the SOSC is satisfied at any point that satisfies the FONC means that

$$\det \overline{H} = \det \left[\begin{matrix} 0 & g_1 & g_2 \\ g_1 & -\lambda g_{11} & -\lambda g_{12} \\ g_2 & -\lambda g_{12} & -\lambda g_{22} \end{matrix}\right]\Bigg|_{(x_1, x_2; \lambda) = (x_1^*, x_2^*; \lambda^*)} < 0 .$$

Note also that the FONC imply $\lambda^* > 0$.

b. The Jacobian of the system of FONC is $\overline{H}$ with factors of -1 in the first row and first column, and thus has the same determinant as $\overline{H}$. Satisfaction of the SOSC therefore implies that the hypotheses of the implicit function theorem are met and the FONC implicitly define x_1^*, x_2^*, and λ^* as differentiable functions of a_1 and a_2 locally.

Differentiating the FONC with respect to a_1 (and using "$(*)$" to denote evaluation at starred values of the arguments):

$$\left[\begin{matrix} 0 & -g_1(*) & -g_2(*) \\ -g_1(*) & -\lambda^* g_{11}(*) & -\lambda^* g_{12}(*) \\ -g_2(*) & -\lambda^* g_{12}(*) & -\lambda^* g_{22}(*) \end{matrix}\right] \left(\begin{matrix} \dfrac{\partial \lambda^*}{\partial a_1} \\ \dfrac{\partial x_1^*}{\partial a_1} \\ \dfrac{\partial x_2^*}{\partial a_1} \end{matrix}\right) = \left(\begin{matrix} 0 \\ -1 \\ 0 \end{matrix}\right).$$

Solving by Cramer's rule:

$$\frac{\partial \lambda^*}{\partial a_1} = \frac{\det \begin{bmatrix} 0 & -g_1(*) & -g_2(*) \\ -1 & -\lambda^* g_{11}(*) & -\lambda^* g_{12}(*) \\ 0 & -\lambda^* g_{12}(*) & -\lambda^* g_{22}(*) \end{bmatrix}}{\det \overline{\overline{H}}} = \frac{\lambda^* \left(g_1(*) g_{22}(*) - g_2(*) g_{12}(*)\right)}{\det \overline{\overline{H}}} > 0 .$$

Differentiating the FONC with respect to a_2:

$$\begin{bmatrix} 0 & -g_1(*) & -g_2(*) \\ -g_1(*) & -\lambda^* g_{11}(*) & -\lambda^* g_{12}(*) \\ -g_2(*) & -\lambda^* g_{12}(*) & -\lambda^* g_{22}(*) \end{bmatrix} \begin{pmatrix} \frac{\partial \lambda^*}{\partial a_2} \\ \frac{\partial x_1^*}{\partial a_2} \\ \frac{\partial x_2^*}{\partial a_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Solving by Cramer's rule:

$$\frac{\partial \lambda^*}{\partial a_2} = \frac{\det \begin{bmatrix} 0 & -g_1(*) & -g_2(*) \\ 0 & -\lambda^* g_{11}(*) & -\lambda^* g_{12}(*) \\ -1 & -\lambda^* g_{12}(*) & -\lambda^* g_{22}(*) \end{bmatrix}}{\det \overline{\overline{H}}} = \frac{-\lambda^* \left(g_1(*) g_{12}(*) - g_2(*) g_{11}(*)\right)}{\det \overline{\overline{H}}} > 0 .$$