Exam #2

Do all four problems. They will be equally weighted. Closed book, open notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

   a. If $F : \mathbb{R}^n \to \mathbb{R}$ is concave, then $g : [0, 1] \to \mathbb{R}$ defined by $g(t) = F(tx)$, where $x \in \mathbb{R}^n$, is concave.

   b. Given that $F : \mathbb{R}^n \to \mathbb{R}$ is strictly quasi-concave and that the set $S$ is defined as:
   
   $S = \{ x \in \mathbb{R}^n : F(x) \geq F(\bar{x}) \text{ for all } \bar{x} \in \mathbb{R}^n \}$. Then $S$ cannot contain more than a single point.

   c. For differentiable $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, consider the problem:
   
   $$\min_{\text{w.r.t. } x, \text{ given } a} F(x; a),$$

   where $x$ is an $n$-dimensional vector of choice variables and $a$ is a scalar parameter. Assume that this problem has a strict global solution for each value of $a$ given by differentiable $x^*(a)$. Define the value function: $F^*(a) = F(x^*(a); a)$. Then, for any given value of $a$, $a = a_0$ say:
   
   $$\frac{d^2}{da^2} F^*(a_0) \geq \frac{\partial^2}{\partial a^2} F(x^*(a_0); a_0)$$

   d. Given differentiable $F : \mathbb{R}^n \to \mathbb{R}$, consider the problem:
   
   $$\max_{\text{w.r.t. } x} F(x) \text{ subject to } x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0.$$  

   If $x^* \in \mathbb{R}_+^n$ is a local solution to this problem, $\frac{\partial F}{\partial x_i}(x^*) = 0$ for all $i = 1, 2, \ldots, n$.

   e. If a bounded sequence in $\mathbb{R}$, $\{ x_n \}_{n=1}^\infty$, has a subsequence that converges to $x \in \mathbb{R}$, then $\{ x_n \}_{n=1}^\infty \to x$. 

2. For $T$ any positive integer, consider the problem:

$$\max_{w,x,t} \left\{ \frac{1}{2} \sum_{t=1}^{T} x_t^{1/2} \right\} \quad \text{subject to} \quad \sum_{t=1}^{r} x_t \leq 1 \quad \text{and} \quad x_t \geq 0 \quad \text{for} \quad t = 1, 2, \ldots, T.$$ 

a. Write down the Kuhn-Tucker conditions for this problem.

b. Solve the problem for the optimal values of the choice variables, $x_t^*$ for $t = 1, 2, \ldots, T$, as functions of $T$.

(Hint: Begin by thinking very carefully about the implications of the Kuhn-Tucker conditions for the general nature of the solution. Will the inequality constraint be binding? Will any of the non-negativity constraints be binding?)

3. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^m$ and let $x \in \mathbb{R}^m$ be such that, for all $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many elements of the sequence. (Notice that $x$ itself need not be an element of the sequence.)

a. For $m = 1$, give an example to show that such a sequence need not converge to $x$.

b. For arbitrary $m$, prove that such a sequence must have a subsequence that converges to $x$.

4. A firm’s production technology is characterized by production function $y = g(x_1, x_2, \ldots, x_n)$ where $y$ denotes the quantity of output and $x_1, x_2, \ldots, x_n$ denote the quantities of the $n$ inputs purchased at fixed prices: $w_1, w_2, \ldots, w_n$. $g(\cdot)$ is differentiable, strictly concave, and has strictly positive marginal products throughout $\mathbb{R}^n_+$. (These assumptions are sufficient to insure that the firm’s cost minimization problem has a regular solution for all positive output and all vectors of positive factor prices.) Find an expression for the slope of the firm’s marginal cost curve and show that it is positive.

(Hint: Start with the constrained optimization problem underlying the firm’s cost function. Differentiate the first-order conditions with respect to $y$. Use the envelope theorem.)