Exam #2
Do all four problems. They will be equally weighted. Closed book, open notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

a. Consider $F : U \rightarrow \mathbb{R}$, where $U$ is a convex subset of $\mathbb{R}^n$. $F(\cdot)$ is concave if and only if $\{x \in U : F(x) \geq a\}$ is convex for all $a \in \mathbb{R}$.

b. Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$, differentiable. Let $x^* \in \mathbb{R}^n$ be a local solution to:

$$\max_{x \in U} F(x) \quad \text{subject to} \quad x_i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n.$$ 

Then, for each $i = 1, 2, \ldots, n$, $\frac{\partial F}{\partial x_i}(x^*) \leq 0$ and $x_i^* \frac{\partial F}{\partial x_i}(x^*) = 0$.

c. Given $F : \mathbb{R}^2 \rightarrow \mathbb{R}$, differentiable, assume that the problem $\min_{x \in \mathbb{R}^2} F(x, a)$, has, for each value of $a$, a strict global solution given by the differentiable function $x^*(a)$. Define the value function: $F^*(a) = F(x^*(a), a)$. Then, for every value of $a = a_0$,

$$\frac{dF^*}{da}(a_0) = \frac{\partial F}{\partial a}(x^*(a_0), a_0) \quad \text{and} \quad \frac{d^2 F^*}{da^2}(a_0) \leq \frac{\partial^2 F}{\partial a^2}(x^*(a_0), a_0).$$

d. If a Cauchy sequence, $\{x_n\}_{n=1}^\infty$, has a convergent subsequence, then $\{x_n\}_{n=1}^\infty$ converges.

e. If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable and has a strict local maximum at $x^* \in \mathbb{R}^n$, then $\frac{\partial^2 F}{\partial x^2}(x^*)$ is negative definite.
2. A consumer purchases and consumes two goods, in non-negative quantities \(x\) and \(y\), to maximize strictly quasi-concave utility, \(U(x, y)\), subject to the constraint that expenditure on the goods be no greater than income \(I > 0\). \(U(\cdot)\) has strictly positive marginal utilities throughout \(\mathbb{R}^2\). Good \(y\) is purchased at price \(p_y > 0\). Purchases of good \(x\) are of two types: \(x = x_1 + x_2\). \(x_1\) units are purchased at a subsidized price of \(p_x - s > 0\), where \(s > 0\). \(x_2\) units are purchased at the "regular" price of \(p_x > 0\). The consumer is limited in the amount of good \(x\) that can be purchased at the subsidized price: \(x_1 \leq \bar{x}\), where \(\bar{x} > 0\) is a constant such that \((p_x - s)\bar{x} < I\).

a. Write down the consumer's utility maximization problem subject to inequality and non-negativity constraints.

b. Write down the Lagrangian for the problem and the Kuhn-Tucker conditions.

c. Denoting the consumer's optimal purchases by \((x_1^*, x_2^*, y^*)\), use the Kuhn-Tucker conditions to establish the following claims. (Note: A simple budget-line-indifference-curve graph will help with the intuition of these results. In each case, the claim is pretty obvious. The problem is to formally derive each one from the Kuhn-Tucker conditions.)

\[
\begin{align*}
  &i. \ x_2^* > 0 \Rightarrow x_1^* = \bar{x}. \\
  &ii. \ x_1^* = \bar{x} \text{ and } x_2^* = 0 \Rightarrow \frac{p_x - s}{p_y} \leq \frac{U_x(x_1^* + x_2^*, y^*)}{U_y(x_1^* + x_2^*, y^*)} \leq \frac{p_x}{p_y}.
\end{align*}
\]

3. Recall the following definitions of the underlined terms:

\(C \subset \mathbb{R}^n\) is open if, for all \(x \in C\), there exists \(\varepsilon > 0\) such that \(B_\varepsilon(x) \subset C\).

A function \(f : \mathbb{R}^n \to \mathbb{R}^m\) is continuous if, for every convergent sequence in \(\mathbb{R}^n\), \(\{x_n\}_{n=1}^\infty \to x \in \mathbb{R}^n\), we have \(\{f(x_n)\}_{n=1}^\infty \to f(x) \in \mathbb{R}^m\).

Use these definitions to prove the following proposition:

Let \(f : \mathbb{R}^n \to \mathbb{R}^m\) be continuous and let \(C \subset \mathbb{R}^m\) be open. Then \(f^{-1}(C) \equiv \{x \in \mathbb{R}^n : f(x) \in C\}\) is open.
4. Consider the problem:

\[
\max_{x_1, x_2} F(x_1, x_2) \text{ subject to } g(x_1, x_2) = b,
\]

where \( F : \mathbb{R}^2 \to \mathbb{R} \) is differentiable and strictly concave, \( g : \mathbb{R}^2 \to \mathbb{R} \) is differentiable and strictly convex, and \( b \) is a scalar parameter. Assume that the problem has a regular solution for \( b = b_0 \), insuring that optimal values of the choice variables exist as differentiable functions of \( b \), \( x^*_1(b) \) and \( x^*_2(b) \), in some neighborhood of \( b_0 \). Assume that \( F(\cdot) \) and \( g(\cdot) \) have strictly positive first partial derivatives at \( x^*(b_0) \equiv (x^*_1(b_0), x^*_2(b_0)) \).

Define the value function: \( F^*(b) \equiv F(x^*(b)) \). Show that:

\[
\frac{dF^*}{db}(b_0) > 0 \quad \text{and} \quad \frac{d^2 F^*}{db^2}(b_0) \leq 0.
\]