Exam #2

Do all four problems. They will be equally weighted. Closed book, open notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

a. Let $U$ be a convex subset of $\mathbb{R}^n$. If $F: U \rightarrow \mathbb{R}$ is quasi-concave then

$$F(hu + (1-h)v) \geq \max\{F(u), F(v)\}$$

for all $u, v \in U$ and for all $h \in [0, 1]$.

b. Given $f : \mathbb{R} \rightarrow \mathbb{R}$ differentiable and $x^* \in \mathbb{R}$. If $f'(x^*) = 0$ and $f''(x^*) < 0$ then $f(\cdot)$ has a strict local maximum at $x^*$.

c. Given differentiable functions $F : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$. Consider the problem:

$$\max_{w.r.t. x} F(x; a) \text{ subject to } g(x) = b,$$

where $x$ is an $n$-dimensional vector of choice variables, $a$ is a scalar parameter that enters the objective function but not the constraint function, and $b$ is a scalar constant. Assume that for $a = a_i$, the problem has a regular solution at $x^*(a_i)$. Define the value function: $F^*(a) \equiv F(x^*(a); a)$. Then $F^*(\cdot)$ is differentiable at $a_i$ and

$$\frac{dF^*}{da}(a_i) = \frac{\partial F}{\partial a}(x^*(a_i); a_i).$$

d. Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable. If $x^* \in \mathbb{R}^n$ is a local solution to

$$\max_{w.r.t. x} F(x) \text{ subject to } x \in \mathbb{R}^n_+,$$

then, for $i = 1, 2, \ldots, n$, $\frac{\partial F}{\partial x_i}(x^*) \geq 0$.

e. Cauchy sequences in $\mathbb{R}^n$ are bounded.
2. Consider three sequences in $\mathbb{R}^1$: \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty, \text{ and } \{z_n\}_{n=1}^\infty$.

For each $n = 1, 2, 3, \ldots$, $x_n \leq y_n \leq z_n$. Also, both \{x_n\}_{n=1}^\infty and \{z_n\}_{n=1}^\infty converge and have the same limit:
\[
\{x_n\}_{n=1}^\infty \rightarrow a \quad \text{and} \quad \{z_n\}_{n=1}^\infty \rightarrow a.
\]

Prove that \{y_n\}_{n=1}^\infty \rightarrow a.

3. $g : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ is differentiable with $\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} > 0$ and $\frac{\partial^2 g}{\partial x^2}$ negative definite throughout $\mathbb{R}^2$. $a_1$ and $a_2$ are positive constants. Consider the problem:
\[
\min_{\text{w.r.t. } x_1, x_2} a_1 x_1 + a_2 x_2 \quad \text{subject to} \quad g(x_1, x_2) = 0.
\]

i. Write down the Lagrangian and the first-order necessary conditions for this problem. Show that the second-order sufficient conditions for a strict local minimum will be satisfied at any solution to the first-order conditions.

ii. Let $x^*_1(a_1, a_2), x^*_2(a_1, a_2)$, and $F^*(a_1, a_2)$ denote the solution values of the choice variables and the value function, respectively. Show that $\frac{\partial F^*}{\partial a_i}$ has the same algebraic sign as $x^*_1$. Sketch and explain a couple of graphs that illustrate the intuition for this result.

iii. Find and sign expressions for $\frac{\partial x^*_1}{\partial a_i}$ and $\frac{\partial x^*_2}{\partial a_i}$.

4. Consider the following utility maximization problem:
\[
\max_{\text{w.r.t. } x_1, x_2} U(x_1, x_2; a, b) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 \leq I, \quad x_1 \geq 0, \quad x_2 \geq 0,
\]

where $x_1$ and $x_2$ are quantities of the two goods, $U(\cdot)$ is the utility function, $a$ and $b$ are positive parameters of the utility function, $p_1$ and $p_2$ are positive prices, and $I$ is positive income. For each of the following utility functions, find restrictions on prices, income, and utility function parameters that are implied by zero consumption of good 1 at the optimum.

i. $U(x_1, x_2; a, b) = (x_1 + a) \cdot (x_2 + b)$.

ii. $U(x_1, x_2; a) = x_2 \cdot \exp(x_1 + a)$. 