Exam #2
Do all four problems. They will be equally weighted. Closed book, open notes. Be sure that your answers are thoroughly explained and presented in a neat and well-organized manner!

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in wording.

a. $U$ is a convex subset of $\mathbb{R}^n$. $F: U \to \mathbb{R}$ is concave if and only if 
$$\{x \in U : F(x) \geq a\} \text{ is convex for all } a \in \mathbb{R}.$$ 

b. If $F: \mathbb{R}^n \to \mathbb{R}$ is differentiable and strictly concave and there exists $x^* \in \mathbb{R}^n$ such that $\frac{\partial F}{\partial x}(x^*) = 0$, then $F(x^*) > F(x)$ for all $x \in \mathbb{R}^n, x \neq x^*$.

c. Given differentiable functions $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Consider the problem:
$$\max_{w,r,f,x} F(x; a) \text{ subject to } g(x;a) = b,$$

where $x$ is an $n$-dimensional vector of choice variables, $a$ is a scalar parameter that enters both the objective function and the constraint function, and $b$ is a scalar constant. Assume that for $a = a_1$, the problem has a regular solution at $x^*(a_1)$. Define the value function: $F^*(a) = F(x^*(a); a)$. Then $F^*(\cdot)$ is differentiable at $a_i$ and
$$\frac{dF^*}{da}(a_i) = \frac{\partial F}{\partial a}(x^*(a_i); a_i).$$

d. If $\{x_n\}_{n=1}^\infty$ is a sequence in a closed subset of $\mathbb{R}^n$, it has a convergent subsequence.

e. If $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^n$ and has a convergent subsequence with limit $x$, then $\{x_n\}_{n=1}^\infty$ converges to $x$. 

2. Consider the following inequality and non-negativity constrained maximization problem:

$$\max_{w, r, t, x, y} 3x + y \quad \text{subject to} \quad x \geq 0, \quad y \geq 0, \quad (x+1)^2 + y^2 \leq 4, \quad x^2 + (y+1)^2 \leq 4.$$ 

(a) Write down the Lagrangian and the Kuhn-Tucker conditions for this problem.

(b) Find values of $x$ and $y$ and any Lagrange multipliers that solve the Kuhn-Tucker conditions.

(Hint: The solution has $x > 0$, $y > 0$, $(x+1)^2 + y^2 = 4$, and $x^2 + (y+1)^2 < 4$.)

(c) Does the solution to the Kuhn-Tucker conditions that you found in part (b) correspond to a solution to the constrained maximization problem? Explain.

3. Let $K_1$ and $K_2$ be compact subsets of $\mathbb{R}^n$. Define: $K = \{x + y : x \in K_1, y \in K_2\}$. Prove that $K$ is closed.

(Hint: Use the Bolzano-Weierstrass theorem twice.)

4. A profit-maximizing firm produces output $y$ using inputs $x_1$ and $x_2$ via the production function $y = f(x_1, x_2)$. The firm faces parametric prices for output and inputs and, to keep things simple, assume that the prices of output and input 2 are both fixed at 1. Denote the price of input 1 by $w_1$. The firm’s profit maximization problem is

$$\max_{w, r, t, x_1, x_2} \pi(w_1; y, x_1, x_2) \quad \text{subject to} \quad y - f(x_1, x_2) = 0,$$

where $\pi(w_1; y, x_1, x_2) = y - w_1 x_1 - x_2$. Assume that, for $w_1$ values in a neighborhood of $w_1 = w_1^0$, the problem has strict global solutions given by differentiable functions of $w_1$:

$$y^*(w_1), x_1^*(w_1), \text{ and } x_2^*(w_1).$$

The firm’s profit function is the value function for this problem:

$$\pi^*(w_1) = \pi(w_1; y^*(w_1), x_1^*(w_1), x_2^*(w_1)) = y^*(w_1) - w_1 x_1^*(w_1) - x_2^*(w_1).$$

Define the “passive profit function” as follows:

$$\pi^p(w_1) = \pi(w_1; y^*(w_1^0), x_1^*(w_1^0), x_2^*(w_1^0)).$$
In other words, the passive profit function shows how the firm’s profit varies as \( w_i \) varies with input and output quantities held fixed at the levels optimal for \( w_i = w_i^0 \).

\( \text{a. Sketch a graph showing the general relationship between } \pi^*\left( \cdot \right) \text{ and } \pi^p\left( \cdot \right) \text{ in a neighborhood of } w_i = w_i^0. \)

(Hint: When the firm can make quantity adjustments in response to price changes, its profit will always be at least as great as when it cannot.)

\( \text{b. Based on the appearance of the graph, use an envelope theorem argument to derive the algebraic sign of } \frac{dx_i^*}{dw_i}(w_i^0). \)