Exam #2

Do all four problems; they will be equally weighted. Closed book, open notes. Your answers must be thoroughly explained and presented in a well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a change in wording.

   a. $U$ is a convex subset of $\mathbb{R}^n$. $F : U \to \mathbb{R}$ is concave. Then the set
      \[ \{(x, y) : x \in U \text{ and } y \leq F(x)\} \]
      is convex in $\mathbb{R}^{n+1}$.

   b. $F : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Suppose that the problem,
      \[ \max_{x_1, x_2} F(x_1, x_2 ; c), \]
      has a regular solution for $c = c_0$ with optimal values of the choice variables given by
      \[ x^*(c) = (x^*_1(c), x^*_2(c)) \]
      for $c$ in a neighborhood of $c_0$. Suppose that \( \frac{\partial^2 F}{\partial x_1 \partial c}(x^*(c_0); c_0) = 0 \).
      Then \( \frac{dx^*_2(c_0)}{dc} \cdot \frac{\partial^2 F}{\partial x_2 \partial c}(x^*(c_0); c_0) \leq 0 \).

   c. $F : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable and $c \in \mathbb{R}$. Suppose the problem
      \[ \max_{w.r.t. x} F(x) \text{ such that } g(x) \leq c \text{ and } x \in \mathbb{R}^n^* \]
      has a local solution, $x^*$, with $g(x^*) < c$. Then
      \[ x^*_i \cdot \frac{\partial F}{\partial x_i}(x^*) = 0 \text{ for all } i = 1, 2, \ldots, n. \]

   d. If a sequence in $\mathbb{R}^m$, \( \{x_n\}_{n=1}^\infty \), has a convergent subsequence with limit $y \in \mathbb{R}^m$, then
      \( \{x_n\}_{n=1}^\infty \to y \).

   e. $S \subset \mathbb{R}^m$ is compact if and only if every sequence in $S$ has a convergent subsequence.
2. a. Prove that the intersection of finitely or infinitely many closed sets in $\mathbb{R}^m$ is closed.

b. Prove that the union of finitely many closed sets in $\mathbb{R}^m$ is closed.

c. Give an example of a union of infinitely many closed sets in $\mathbb{R}^1$ that is not closed.

3. A sum of $C$ dollars is available for allocation among $n$ investment projects. If the non-negative amount $x_i$ is allocated to project $i$, for $i = 1, 2, \ldots, n$, the expected return from the portfolio will be $\sum_{i=1}^{n} (\alpha_i x_i - \beta_i \exp(x_i))$, where the $\alpha_i$s and $\beta_i$s are constants satisfying $\alpha_i > \beta_i > 0$ for all $i = 1, 2, \ldots, n$. The problem is to choose the $x_i$s so as to maximize expected return subject to the constraint that each $x_i$ be non-negative and that the sum of the $x_i$s be no more than $C$.

a. Write down the Kuhn-Tucker conditions for this problem.

b. If there is a solution to the Kuhn Tucker conditions, is it necessarily a global solution to the problem? Explain.

c. Using $x_i^*$ to denote the optimal value of $x_i$, use the Kuhn-Tucker conditions to show that:

i. $\sum_{i=1}^{n} x_i^* < C \Rightarrow \sum_{i=1}^{n} \ln \left( \frac{\alpha_i}{\beta_i} \right) < C$

ii. $x_j^* > 0$ and $x_k^* = 0 \Rightarrow \alpha_k - \beta_k < \alpha_j - \beta_j$.

4. A firm uses three inputs to produce output. For any output level, the firm chooses the cost-minimizing input combination. In the short-run, only two of the inputs are variable while one, let’s call it “plant,” is fixed. In the long run, all three inputs are variable. Let’s use the term “$Q_0$-optimal plant” to mean the plant size that is optimal for output level $Q_0$ in the long run. Show that, at output level $Q_0$, the slope of short-run marginal cost in the $Q_0$-optimal plant is at least as great as the slope of long-run marginal cost.

(Hint: Sketch a graph of short-run cost in the $Q_0$-optimal plant and long-run cost, both as functions of output, in a neighborhood of $Q_0$, and make an argument based on the appearance of the graph. You will first have to formalize this problem in terms of mathematical notation. In doing so, you must thoroughly explain any notation that you introduce.)