Exam #2 Solution Outline

1. a. True

b. False. Change "for all $u, v \in \mathbb{R}^n$" to "for all $u, v \in \mathbb{R}^n$ such that $F(u) \geq F(v)$."

c. False. Change "strict global maximum" to "strict local maximum."

d. False. Change "closed" to "compact" (closed and bounded).

e. True.

2. For $k = 1, 2, \ldots, m$, let $\{x_{kn}\}_{n=1}^\infty$ denote the sequence of $k$th elements of the vectors in $\{x_n\}_{n=1}^\infty$. The claim is $\{x_n\}_{n=1}^\infty$ is Cauchy in $\mathbb{R}^m$ $\Leftrightarrow \{x_{kn}\}_{n=1}^\infty$ is Cauchy in $\mathbb{R}^1$ for each $k = 1, 2, \ldots, m$.

Proof: ($\Rightarrow$) For given $k$ and given $\varepsilon > 0$, want to show that there exists $N$ such that $i, j \geq N \Rightarrow \|x_{ki} - x_{kj}\| = |x_{ki} - x_{kj}| < \varepsilon$.

Take $N$ such that $i, j \geq N \Rightarrow \|x_i - x_j\| < \varepsilon$ (Can do this because $\{x_n\}_{n=1}^\infty$ is Cauchy.)

Then for $i, j \geq N$, $|x_{ki} - x_{kj}| = \left(\sum_{l=1}^{m} (x_{kl} - x_{kj})^2\right)^{1/2} \leq \left(\sum_{l=1}^{m} (x_{kl} - x_{kj})^2\right)^{1/2} = \|x_i - x_j\| < \varepsilon$ Q.E.D.

Proof: ($\Leftarrow$) For given $\varepsilon > 0$, want to show that there exists $N$ such that $i, j \geq N \Rightarrow \|x_i - x_j\| < \varepsilon$.

For each $k = 1, 2, \ldots, m$, take $N_k$ such that $i, j \geq N_k \Rightarrow \|x_{ki} - x_{kj}\| = |x_{ki} - x_{kj}| < \varepsilon/\sqrt{m}$ (Can do this because each $\{x_{kn}\}_{n=1}^\infty$ is Cauchy.)

Take $N = \max\{N_1, N_2, \ldots, N_k\}$ Then, for $i, j \geq N$, $\|x_i - x_j\| = \left(\sum_{k=1}^{m} (x_{ki} - x_{kj})^2\right)^{1/2} < \left(\sum_{k=1}^{m} \frac{\varepsilon^2}{m}\right)^{1/2} = \varepsilon$. Q.E.D.
3. a. \[ \max_{w,r,t,x_1,x_2,x_3} \ln(x_1 + a) + b \ln x_2 + c \ln x_3 \text{ subject to } p_1x_1 + p_2x_2 + p_3x_3 \leq I; \ x_1, \ x_2, \ x_3 \geq 0 \]

Lagrangian: \[ L(x_1, x_2, x_3; \lambda) = \ln(x_1 + a) + b \ln x_2 + c \ln x_3 + \lambda(I - p_1x_1 - p_2x_2 - p_3x_3) \]

Kuhn-Tucker conditions:

(1) \[ \frac{\partial L(x^*; \lambda^*)}{\partial x_1} = \frac{1}{x_1 + a} - p_1\lambda^* \leq 0, \quad x_1^* \geq 0, \quad x_1^* \frac{\partial L(x^*; \lambda^*)}{\partial x_1} = 0 \]

(2) \[ \frac{\partial L(x^*; \lambda^*)}{\partial x_2} = \frac{b}{x_2} - p_2\lambda^* \leq 0, \quad x_2^* \geq 0, \quad x_2^* \frac{\partial L(x^*; \lambda^*)}{\partial x_2} = 0 \]

(3) \[ \frac{\partial L(x^*; \lambda^*)}{\partial x_3} = \frac{c}{x_3} - p_3\lambda^* \leq 0, \quad x_3^* \geq 0, \quad x_3^* \frac{\partial L(x^*; \lambda^*)}{\partial x_3} = 0 \]

(4) \[ \frac{\partial L(x^*; \lambda^*)}{\partial \lambda} = I - p_1x_1^* - p_2x_2^* - p_3x_3^* \geq 0, \quad \lambda^* \geq 0, \quad \lambda^* \frac{\partial L(x^*; \lambda^*)}{\partial \lambda} = 0 \]

b. The marginal conditions in (2) and (3) could not hold with \[ x_1^* = 0 \] or \[ x_2^* = 0 \]. So we must have the non-negativity constraints slack in these two cases: \[ x_1^* > 0, x_2^* > 0 \]. But then the marginal conditions in (2) and (3) (which must hold as equalities) require \[ \lambda^* > 0 \]. Complementary slackness in (4) then tells us that the budget constraint must be binding: \[ p_1x_1^* + p_2x_2^* + p_3x_3^* = I \]. Thus there are only two solution types and they are distinguished by whether the non-negativity constraint on \[ x_1 \] is binding or slack: \[ x_1^* = 0 \] or \[ x_1^* > 0 \].

c. Case 1.: \[ x_1^* = 0 \]. From the marginal conditions in (2) and (3): \[ x_2^* = \frac{b}{p_2\lambda^*} \] and \[ x_3^* = \frac{c}{p_3\lambda^*} \]. Substituting into the budget constraint (along with \[ x_1^* = 0 \]) and solving yields: \[ \lambda^* = \frac{b + c}{I} \]. Substituting this into the above expressions for \[ x_2^* \] and \[ x_3^* \] yields demand equations:

\[ x_2^* = \frac{bI}{p_2(b + c)} \] and \[ x_3^* = \frac{cI}{p_3(b + c)} \].

To have a solution of this type, the marginal condition in (1) must be satisfied with
\[ x_i^* = 0, \text{ and this requires } \frac{1}{a} = \frac{p_i (b + c)}{I} \leq 0 \text{ or } p_i \geq \frac{I}{a(b + c)}. \]

Case 2: \( x_i^* > 0 \). The marginal conditions in (1), (2), and (3) all hold as equalities.

Solving: \( x_1^* = \frac{1}{p_1 \lambda^*} - a, \ x_2^* = \frac{b}{p_2 \lambda^*}, \text{ and } x_3^* = \frac{c}{p_3 \lambda^*}. \) Substituting into the budget constraint (which also holds as an equality) and solving: \( \lambda^* = \frac{1 + b + c}{I + p_i a}. \) Substituting into the expressions above gives demand equations:

\[ x_1^* = \frac{I + p_i a}{p_1(1 + b + c)} - a, \ x_2^* = \frac{b(I + p_i a)}{p_2(1 + b + c)}, \text{ and } x_3^* = \frac{c(I + p_i a)}{p_3(1 + b + c)}. \]

Consistency with \( x_i^* > 0 \) requires \( p_i < \frac{I}{a(b + c)}. \)

4. The Lagrangian is \( L(x_1, x_2; \lambda) = f(x_1, x_2; \alpha) + \lambda (b - g(x_1, x_2)). \) Optimal \( x_1^*, x_2^* \) must satisfy the first order conditions:

\[ \frac{\partial L}{\partial \lambda} (\ast) = b - g(x_1^*, x_2^*) = 0 \]

\[ \frac{\partial L}{\partial x_1} (\ast) = \frac{\partial f}{\partial x_1}(x_1^*, x_2^*; \alpha) - \lambda^* \frac{\partial g}{\partial x_1}(x_1^*, x_2^*) = 0 \]

\[ \frac{\partial L}{\partial x_2} (\ast) = \frac{\partial f}{\partial x_2}(x_1^*, x_2^*; \alpha) - \lambda^* \frac{\partial g}{\partial x_2}(x_1^*, x_2^*) = 0 \]

(As noted in the problem, when \( \alpha = \alpha_0, \frac{\partial g}{\partial x_2} (\ast) \neq 0 \) so the constraint qualification is satisfied and the first-order conditions are, in fact, necessary.) The Jacobian for this system is:

\[
J = \begin{bmatrix}
0 & -\frac{\partial g}{\partial x_1} (\ast) & -\frac{\partial g}{\partial x_2} (\ast) \\
-\frac{\partial g}{\partial x_1} (\ast) & \frac{\partial^2 L}{\partial x_1^2} (\ast) & \frac{\partial^2 L}{\partial x_1 \partial x_2} (\ast) \\
-\frac{\partial g}{\partial x_2} (\ast) & \frac{\partial^2 L}{\partial x_1 \partial x_2} (\ast) & \frac{\partial^2 L}{\partial x_2^2} (\ast)
\end{bmatrix}
\]
This is the same as the bordered Hessian matrix but with the first row and first column multiplied by -1. Thus, the determinant of the Jacobian is the same as the determinant of the bordered Hessian; positive, by the SOSC. So, by the implicit function theorem, the first-order conditions define differentiable functions: \( x_1^*(\alpha), x_2^*(\alpha), \) and \( \lambda^*(\alpha) \).

Differentiating the first order conditions and evaluating at \( \alpha = \alpha_0 \), we have:

\[
J \left( \frac{\partial \lambda^*}{\partial \alpha}(\alpha_0), \frac{\partial x_1^*}{\partial \alpha}(\alpha_0), \frac{\partial x_2^*}{\partial \alpha}(\alpha_0) \right) = \begin{pmatrix}
-\frac{\partial^2 L}{\partial \lambda \partial \alpha}(\alpha_0) & 0 \\
-\frac{\partial^2 L}{\partial x_1 \partial \alpha}(\alpha_0) & \frac{\partial^2 f}{\partial x_1 \partial \alpha}(\alpha_0) \\
-\frac{\partial^2 L}{\partial x_2 \partial \alpha}(\alpha_0) & 0
\end{pmatrix}
= \begin{pmatrix}
0 \\
-\frac{\partial^2 f}{\partial x_1 \partial \alpha}(\alpha_0) \\
0
\end{pmatrix}.
\]

(Using the fact that \( \frac{\partial^2 f}{\partial x_2 \partial \alpha}(x_1^*(\alpha_0), x_2^*(\alpha_0), \lambda^*(\alpha_0)) = 0 \).)

Solving by Cramer's rule:

\[
\frac{\partial x_1^*}{\partial \alpha}(\alpha_0) = \frac{\begin{vmatrix}
0 & 0 & -\frac{\partial g}{\partial x_2}(\alpha_0) \\
-\frac{\partial g}{\partial x_1}(\alpha_0) & -\frac{\partial^2 f}{\partial x_1 \partial \alpha}(\alpha_0) & \frac{\partial^2 L}{\partial x_1 \partial \alpha}(\alpha_0) \\
-\frac{\partial g}{\partial x_2}(\alpha_0) & 0 & \frac{\partial^2 L}{\partial x_2 \partial \alpha}(\alpha_0)
\end{vmatrix}}{\det J}
\]

\[
\frac{\partial x_2^*}{\partial \alpha}(\alpha_0) = \frac{\left(\frac{\partial^2 f}{\partial x_2 \partial \alpha}(\alpha_0)\right)^2 - \left(\frac{\partial g}{\partial x_2}(\alpha_0)\right)^2}{\det J}
\]

Since \( \det J > 0 \), \( \frac{\partial x_1^*}{\partial \alpha}(\alpha_0) \) and \( \frac{\partial^2 f}{\partial x_2 \partial \alpha}(x_1^*(\alpha_0), x_2^*(\alpha_0), \lambda^*(\alpha_0)) \) have the same sign.