Exam #2 Solution Outline

1. a. **False.** Change “max” to “min”.  b. **True.**  c. **True.**

   d. **False.** Change “≥” to “≤”.  e. **True.**

2. Proof:

   Given $\varepsilon > 0$, want to show that there exists $N$ such that $n \geq N \Rightarrow \left| y_n - a \right| < \varepsilon$.

   By convergence of $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$,

   there exists $N_1$ such that $n \geq N_1 \Rightarrow \left| x_n - a \right| < \varepsilon \Rightarrow -\varepsilon < x_n - a < \varepsilon \quad (1)$

   and there exists $N_2$ such that $n \geq N_2 \Rightarrow \left| z_n - a \right| < \varepsilon \Rightarrow -\varepsilon < z_n - a < \varepsilon \quad (2)$

   Take $N = \max\{N_1, N_2\}$.

   For all $n$: $x_n - a \leq y_n - a \leq z_n - a$.

   Using (1) and (2), for $n \geq N$: $-\varepsilon < x_n - a \leq y_n - a \leq z_n - a < \varepsilon$

   $\Rightarrow \left| y_n - a \right| < \varepsilon$.  **Q.E.D.**
3. i. Lagrangian: \( L(x_1, x_2; \lambda; a_1, a_2) = a_1 x_1 + a_2 x_2 + \lambda (-g(x_1, x_2)) \).

FONC:
\[
\frac{\partial L}{\partial \lambda} = -g(x_1, x_2) = 0 \\
\frac{\partial L}{\partial x_1} = a_1 - \lambda g_1(x_1, x_2) = 0 \\
\frac{\partial L}{\partial x_2} = a_2 - \lambda g_2(x_1, x_2) = 0
\]

where subscripts on \( g() \) denote partial derivatives.

The bordered Hessian is:
\[
\begin{bmatrix}
0 & g_1 & g_2 \\
g_1 & -\lambda g_{11} & -\lambda g_{12} \\
g_2 & -\lambda g_{12} & -\lambda g_{22}
\end{bmatrix}, \text{ and the second-order sufficient condition is } \det \bar{H} < 0.
\]

\[
\det \bar{H} = \lambda \left( g_{11} g_2^2 - 2g_{12} g_1 g_2 + g_{22} g_1^2 \right) = \lambda \left( -g_2, g_1 \right) \begin{bmatrix}
g_{11} & g_{12} \\
g_{12} & g_{22}
\end{bmatrix} \begin{bmatrix}
g_2 \\
g_1
\end{bmatrix}.
\]

From the FONC and \( a_1, a_2, g_1, g_2 > 0 \), we have that \( \lambda > 0 \). A negative definite Hessian for \( g() \) means that the quadratic form in the above expression is negative and, thus, \( \det \bar{H} < 0 \), satisfying the SOSC.

ii. By the envelope theorem:
\[
\frac{\partial F^*}{\partial a_1} = \frac{\partial}{\partial a_1} L(x_1^*, x_2^*; \lambda^*; a_1, a_2) = x_1^*.
\]

The negative definite Hessian throughout \( \mathcal{R}^2 \) means that \( g() \) is strictly concave and therefore strictly quasi-concave. So the level curves of the \( g() \) function bound convex upper contour sets. With \( g_1, g_2 > 0 \), the direction of increasing values of \( g() \) is to the "northeast" in the \( x_1, x_2 \) plane. \( a_1, a_2 > 0 \) mean that the level curves of the linear objective function have negative slope equal to \( a_1/a_2 \) in absolute value, and that the direction of increasing values of the objective function is also to the "northeast." When \( a_1 \) increases, the level curve of the objective function associated with a given value becomes steeper, rotating about a fixed point on the \( x_2 \) axis. The graphs on the last page show that when \( x_1^* \) is positive, this increase in \( a_1 \) forces the solution onto a worse
(higher-value) level curve of the objective function. When \( x_1^* \) is negative, an increase in \( a_1 \) enables attainment of a better (lower-value) level curve of the objective function.

**iii.** Differentiating the FONC:

\[
\begin{bmatrix}
0 & -g_1 & -g_2 \\
-g_1 & -\lambda g_{11} & -\lambda g_{12} \\
-g_2 & -\lambda g_{12} & -\lambda g_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \lambda^*}{\partial a_1} \\
\frac{\partial \lambda^*}{\partial x_1^*} \\
\frac{\partial \lambda^*}{\partial x_2^*}
\end{bmatrix}
= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.
\]

Note that the 3 x 3 matrix on the right is \( H \) with factors of -1 in the first row and first column, and therefore has the same determinant as \( H \).

Solving by Cramer's Rule:

\[
\frac{\partial x_1^*}{\partial a_1} = \frac{1}{\det H} \cdot \det \begin{bmatrix} 0 & 0 & -g_2 \\ -g_1 & -1 & -\lambda g_{12} \\ -g_2 & 0 & -\lambda g_{22} \end{bmatrix} = \frac{g_2^2}{\det H} < 0
\]

\[
\frac{\partial x_2^*}{\partial a_1} = \frac{1}{\det H} \cdot \det \begin{bmatrix} 0 & -g_1 & 0 \\ -g_1 & -\lambda g_{11} & -1 \\ -g_2 & -\lambda g_{12} & 0 \end{bmatrix} = \frac{-g_1 g_2}{\det H} > 0.
\]

**4. i.** \( \max \ (x_1 + a)(x_2 + b) \) subject to \( p_1 x_1 + p_2 x_2 \leq I, \ x_1 \geq 0, \ x_2 \geq 0 \)

\[
L = (x_1 + a)(x_2 + b) + \lambda (I - p_1 x_1 - p_2 x_2)
\]

Kuhn-Tucker conditions (in each case, "c.s." refers to complementary slackness):

1. \( \frac{\partial L}{\partial x_1} = x_1^* + b - \lambda^* p_1 \leq 0, \ x_1^* \geq 0, \ \text{and c.s.} \)

2. \( \frac{\partial L}{\partial x_2} = x_1^* + a - \lambda^* p_2 \leq 0, \ x_2^* \geq 0, \ \text{and c.s.} \)

3. \( \frac{\partial L}{\partial \lambda} = I - p_1 x_1^* - p_2 x_2^* \geq 0, \ \lambda^* \geq 0, \ \text{and c.s.} \)
Marginal conditions in (1) or (2) \( \Rightarrow \lambda^* > 0 \). Then c.s. in (3) \( \Rightarrow p_1x_1^* + p_2x_2^* = l \).

If \( x_1^* \) were zero, we would have \( x_2^* = \frac{l}{p_2} > 0 \), so the marginal condition in (2) would hold with equality \( \Rightarrow \lambda^* = \frac{a}{p_2} \). Substituting into the marginal condition in (1):

\[
\frac{l}{p_2} + b - p_1 \frac{a}{p_2} \leq 0 \quad \text{or} \quad p_1 \geq \frac{l}{a} + \frac{b}{p_2} \quad \text{or} \quad l \leq p_1a - p_2b.
\]

ii. \( \max w.r.t. x_1, x_2 \) subject to \( p_1x_1 + p_2x_2 \leq l, \quad x_1 \geq 0, \quad x_2 \geq 0 \)

\[
L = x_2 \exp(x_1 + a) + \lambda (l - p_1x_1 - p_2x_2)
\]

Kuhn-Tucker conditions:

1. \( \frac{\partial L}{\partial x_1} = x_2^* \exp(x_1^* + a) - \lambda^* p_1 \leq 0, \quad x_1^* \geq 0, \quad \text{and c.s.} \)

2. \( \frac{\partial L}{\partial x_2} = \exp(x_1^* + a) - \lambda^* p_2 \leq 0, \quad x_2^* \geq 0, \quad \text{and c.s.} \)

3. \( \frac{\partial L}{\partial \lambda} = l - p_1x_1^* - p_2x_2^* \geq 0, \quad \lambda^* \geq 0, \quad \text{and c.s.} \)

The marginal condition in (2) \( \Rightarrow \lambda^* > 0 \). Then c.s. in (3) \( \Rightarrow p_1x_1^* + p_2x_2^* = l \).

If \( x_1^* \) were zero, we would have \( x_2^* = \frac{l}{p_2} > 0 \), so the marginal condition in (2) would hold with equality \( \Rightarrow \lambda^* = \frac{\exp(a)}{p_2} \). Substituting into the marginal condition in (1):

\[
\frac{l}{p_2} \exp(a) - \frac{\exp(a)}{p_2} p_1 \leq 0 \quad \text{or} \quad p_1 \geq l.
\]
Graphs for question 3 ii: When $a_1$ increases the level curve of the objective function that is associated with the value of $F^*$ at the initial optimum rotates from $L_1$ to $L_2$, becoming steeper. The new optimum occurs on level curve $L_3$.

$L_3$ corresponds to a higher value of the objective function in this case.

$L_3$ corresponds to a lower value of the objective function in this case.