Econ 600
Schroeter
Fall 2011

Exam #2 Solution Outline

1. *a*. I intended the answer to this one to be **True**. But the conclusion actually requires that $\lambda^* \neq 0$. So I gave credit either for **True** or for **False** with the correction noted.

b. **False.** Change "quasi-concave" to "concave."

c. **False.** Change "$\det \frac{\partial^2 F}{\partial x^2}\left(x^*\right) < 0$" to "$(-1)^n \det \frac{\partial^2 F}{\partial x^2}\left(x^*\right) > 0$." The fact that the matrix in question is a Hessian is irrelevant. The point is that necessary and sufficient conditions for negative definiteness of a matrix are that the leading principal minors alternate in sign beginning with negative.

d. **True.** This is just the familiar envelope theorem result.

e. **True.** A result proved in lecture is that every bounded monotone sequence of reals converges. And we also know that convergent sequences are Cauchy.

2. Proof: Let $\{y_n\}_{n=1}^{\infty}$ be a convergent sequence in C with limit y. Want to show: $y \in C$.

Construct $\{x_n\}_{n=1}^{\infty}$ in B such that $y_n = f(x_n)$ for all $n = 1, 2, \ldots$

Because B is compact, the Bolzano-Weierstrass theorem implies that $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence:

$\{x_{n_k}\}_{k=1}^{\infty} \to x \in B$ (because B is closed). $f(\cdot)$ continuous implies:

$\{y_{n_k}\}_{k=1}^{\infty} = \{f(x_{n_k})\}_{k=1}^{\infty} \to f(x) \in C$.

It was given that $\{y_n\}_{n=1}^{\infty} \to y$. Now we also have that a subsequence, $\{y_{n_k}\}_{k=1}^{\infty} \to f(x)$. Subsequences of convergent sequences all converge and have the same limit.

(It was fine to simply assert this. But, to supply a proof:

> Suppose $y \neq f(x)$. Then there exists $\varepsilon > 0$ such that $|y - f(x)| > \varepsilon$.
>
> $\{y_n\}_{n=1}^{\infty} \to y \Rightarrow$ there exists N such that $n \geq N \Rightarrow |y_n - y| < \varepsilon/2$

$\{y_{n_k}\}_{k=1}^{\infty} \to f(x) \Rightarrow$ there exists K such that $k \geq K \Rightarrow \left| y_{n_k} - f(x) \right| < \varepsilon/2$

Take $N^* = \max\{N, n_K\}$.

$$\begin{aligned} | y - f(x) | &= \left| y - y_{N^*} + y_{N^*} - f(x) \right| \\ &\leq \left| y_{N^*} - y \right| + \left| y_{N^*} - f(x) \right| \quad \text{(by the triangle inequality)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

A contradiction. Therefore $y = f(x)$.)

Thus $y = f(x) \in C$ *Q.E.D.*

3. Proof: Given $c_1, c_2 \in \Re$ *and* $h \in [0, 1]$, want to show:

$$F^*(hc_1 + (1-h)c_2) \geq hF^*(c_1) + (1-h)F^*(c_2).$$

Let $\hat{x} = hx^*(c_1) + (1-h)x^*(c_2)$.

$$g(\hat{x}) \leq hg(x^*(c_1)) + (1-h)g(x^*(c_2)) \quad \text{(by convexity of } g(\cdot))$$

$$\leq hc_1 + (1-h)c_2$$

(Because, for $c = c_1, c_2$, $x^*(c)$ must satisfy the constraint. Therefore, $g(x^*(c_1)) \leq c_1$ and $g(x^*(c_2)) \leq c_2$.)

Because $\hat{x}$ satisfies the constraint: $g(\hat{x}) \leq hc_1 + (1-h)c_2$,

and $x^*(hc_1 + (1-h)c_2) = \underset{w.r.t.\ x}{\arg\max}\{F(x) \text{ subject to } g(x) \leq hc_1 + (1-h)c_2\}$, we have

$$F(\hat{x}) \leq F(x^*(hc_1 + (1-h)c_2)) = F^*(hc_1 + (1-h)c_2). \qquad (*)$$

Also, $F(\hat{x}) \geq hF(x^*(c_1)) + (1-h)F(x^*(c_2))$ (by concavity of $F(\cdot)$)

$$= hF^*(c_1) + (1-h)F^*(c_2) \qquad (**)$$

Combining (*) and (**): $F^*(hc_1 + (1-h)c_2) \geq hF^*(c_1) + (1-h)F^*(c_2)$ *Q.E.D.*

4\. *a*. $L = \ln x_1 + \ln x_2 + \ln x_3 + \lambda\left(R - ax_1 - bx_2 - cx_3\right) + \mu\left(S - x_1 - x_2\right)$

(*i*.) $\frac{1}{x_1} - a\lambda - \mu \le 0, \quad x_1 \ge 0, \quad \left(\frac{1}{x_1} - a\lambda - \mu\right)x_1 = 0$

(*ii*.) $\frac{1}{x_2} - b\lambda - \mu \le 0, \quad x_2 \ge 0, \quad \left(\frac{1}{x_2} - b\lambda - \mu\right)x_2 = 0$

(*iii*.) $\frac{1}{x_3} - c\lambda \le 0, \quad x_3 \ge 0, \quad \left(\frac{1}{x_3} - c\lambda\right)x_3 = 0$

(*iv*.) $R - ax_1 - bx_2 - cx_3 \ge 0, \quad \lambda \ge 0, \quad \left(R - ax_1 - bx_2 - cx_3\right)\lambda = 0$

(*v*.) $S - x_1 - x_2 \ge 0, \quad \mu \ge 0, \quad \left(S - x_1 - x_2\right)\mu = 0$

b. The marginal conditions in (*i*.), (*ii*.), and (*iii*.) imply that $x_1, x_2, x_3 > 0$. Complementary slackness then requires that these marginal conditions hold as equalities. The marginal condition in (*iii*.) further implies that $\lambda > 0$. So complementary slackness in (*iv*.) implies:

$$ax_1 + bx_2 + cx_3 = R. \tag{*}$$

Assume that $x_1 + x_2 < S$. Then complementary slackness in (*v*.) implies that $\mu = 0$. Making this substitution in the marginal conditions in (*i*.), (*ii*.), and (*iii*.) (as equalities) and solving them simultaneously with (*) yields:

$$\lambda = \frac{3}{R}, \quad x_1 = \frac{R}{3a}, \quad x_2 = \frac{R}{3b}, \quad \text{and} \quad x_3 = \frac{R}{3c}.$$

In order to be consistent with the assumption that the second inequality constraint is slack, we need

$$x_1 + x_2 = \frac{R}{3a} + \frac{R}{3b} < S,$$

which implies: $R < \frac{3ab}{a+b}S$.