Exam #1

Do all three problems. Weights: #1 - 30%, #2 - 35%, #3 - 35%. Closed book, closed notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in the wording.

a. Given $F : \mathbb{R}^n \to \mathbb{R}$ differentiable. If there exists $x^* \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $F(x^*) \geq F(x)$ for all $x \in B_\varepsilon (x^*)$ then

$$\frac{\partial^2 F}{\partial x^2}(x^*)$$

is negative semi-definite.

b. Given $F : \mathbb{R}^n \to \mathbb{R}$ differentiable. $F(\cdot)$ is strictly concave if and only if

$$F(x) < F(x^*) + \frac{\partial F}{\partial x}(x^*)(x-x^*)$$

for all $x, x^* \in \mathbb{R}^n$, $x \neq x^*$.

c. Given $F : \mathbb{R}^n \to \mathbb{R}$ differentiable; $g \in \mathbb{R}^n$, a $1 \times n$ vector with $g \neq 0$; and $b$ a scalar constant. Consider the problem:

$$\max_{w.r.t. x} F(x) \text{ subject to } g \cdot x = b, \quad (*)$$

and define the Lagrangian:

$$L(x; \lambda) = F(x) + \lambda(b - g \cdot x).$$

If $F(\cdot)$ is quasi-concave and $x^* \in \mathbb{R}^n$ and $\lambda^* \neq 0$ are such that $(x^*; \lambda^*)$ is a stationary point of $L(\cdot)$, then $x^*$ is a global solution to problem $(*)$.

d. Given $F : \mathbb{R}^2 \to \mathbb{R}$ differentiable and $x^* \in \mathbb{R}^2$ with nonzero gradient vector, $\frac{\partial F}{\partial x}(x^*)$.

Then $\frac{\partial F}{\partial x}(x^*)$ is tangent to the level curve of $F(\cdot)$ through $x^*$.

e. Define $F(x_1, x_2) = 1 - x_1^2 - x_2^2$. At the point $(x_1, x_2) = (0, 0)$, the second order sufficient condition for a strict local maximum is not satisfied.
2. $F : \mathbb{R}^n \to \mathbb{R}$ is strictly quasi-concave. Suppose that $x^* \in \mathbb{R}^n$ is a local solution to

$$\max_{w.r.t. x} F(x)$$

Prove that $x^*$ is a strict global solution.

3. A firm uses $n$ inputs in quantities $x_1, x_2, \ldots, x_n$ to produce output via the differentiable production function $q = F(x_1, x_2, \ldots, x_n)$. The firm faces parametric prices for output $(p)$ and for each of the $n$ inputs $(w_1, w_2, \ldots, w_n)$. Assume that, for every vector of strictly positive prices, profit maximizing employment levels of the $n$ inputs exist as differentiable functions of prices:

$$x_i^* (p; w_1, w_2, \ldots, w_n) \text{ for } i = 1, 2, \ldots, n.$$ 

Show that, for all $i \neq j = 1, 2, \ldots, n$,

$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}.$$ 

(Hint: Use the envelope theorem and Young's theorem. Young's theorem says that the second-order cross-partial derivatives of a differentiable function are equal, regardless of the order in which the derivatives are taken.)