Exam #1

Do all three problems. Weights: #1 - 30%, #2 - 35%, #3 - 35%. Closed book, closed notes. Be sure your answers are presented in a neat and well-organized manner.

1. Answer True or False for each of the following statements. If the statement is false, indicate how it could be changed to a true statement with a small change in the wording.

a. If \( F : \mathbb{R} \to \mathbb{R} \) is differentiable and concave then, for all \( u, v \in \mathbb{R} \),

\[
F(u) \leq F(v) + F'(v)(u - v).
\]

b. Given \( F : \mathbb{R}^n \to \mathbb{R} \) differentiable. If \( x^* \in \mathbb{R}^n \) is such that

\[
\frac{\partial F}{\partial x}(x^*) = 0 \quad \text{and} \quad \frac{\partial^2 F}{\partial x^2}(x^*) \text{ is negative semi-definite}
\]

then \( x^* \) is a local maximum of \( F(\cdot) \).

c. Given \( F : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \), differentiable, and \( b \in \mathbb{R} \), a constant. If \( x^* \in \mathbb{R}^n \) is a local solution to \( \max_{w.r.t. x} F(x) \) subject to \( g(x) = b \), and if \( \frac{\partial g}{\partial x}(x^*) \neq 0 \), then there exists \( \lambda^* \in \mathbb{R} \) such that

\[
\frac{\partial F}{\partial x}(x^*) = \lambda^* \frac{\partial g}{\partial x}(x^*).
\]

d. Given \( F : \mathbb{R}^2 \to \mathbb{R} \), differentiable, assume that the problem \( \max_{w.r.t. x} F(x; a) \) has, for each value of \( a \), a strict global solution given by the differentiable function \( x^*(a) \). Define the value function: \( F^*(a) \equiv F(x^*(a), a) \). Then, for every value of \( a = a_0 \),

\[
\frac{d F^*(a_0)}{da}(a_0) = \frac{\partial F}{\partial a}(x^*(a_0), a_0) \quad \text{and} \quad \frac{d^2 F^*(a_0)}{da^2}(a_0) \leq \frac{\partial^2 F}{\partial a^2}(x^*(a_0), a_0).
\]
e. Given $F : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$, differentiable, and $b \in \mathbb{R}$, a constant. If $x^* \in \mathbb{R}^2$ is a local solution to $\max_{u \in F(x)} F(x)$ subject to $g(x) = b$, and if $\frac{\partial g}{\partial x}(x^*) \neq 0$, then

$$\det \begin{bmatrix}
0 & \frac{\partial g}{\partial x_1}(x^*) & \frac{\partial g}{\partial x_2}(x^*) \\
\frac{\partial g}{\partial x_1}(x^*) & \frac{\partial^2 F}{\partial x_1^2}(x^*) & \frac{\partial^2 F}{\partial x_1 \partial x_2}(x^*) \\
\frac{\partial g}{\partial x_2}(x^*) & \frac{\partial^2 F}{\partial x_2^2}(x^*) & \frac{\partial^2 F}{\partial x_2^2}(x^*)
\end{bmatrix} \geq 0.$$ 

2. Let $U$ be a convex subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be quasi-concave and let $g : \mathbb{R} \to \mathbb{R}$ be increasing. (That is, for all $r_1, r_2 \in \mathbb{R}$, $r_1 \geq r_2 \Rightarrow g(r_1) \geq g(r_2)$.) Define $G : U \to \mathbb{R}$ by $G(u) = g(f(u))$. Prove that $G(\cdot)$ is quasi-concave.

3. A firm uses one input in quantity $x$ to produce two outputs in quantities $y_1$ and $y_2$. Technology is summarized by the input requirement function: $x = f(y_1, y_2)$. The firm faces parametric prices for the input and the two outputs with the price of the input fixed at 1 and the prices of the outputs denoted $p_1$ and $p_2$, for $y_1$ and $y_2$, respectively.

Consider the problem of maximizing profit by choice of the two output quantities. Suppose that, for a given initial value of the output price vector, the problem has a regular solution at which the profit maximizing output levels are locally differentiable functions of prices: $y_i^*(p_1, p_2)$, for $i = 1$ and 2. (Note: A "regular" solution means a solution at which the second-order sufficient conditions are satisfied.) Show that:

$$\left( \frac{\partial y_1^*}{\partial p_1} \right) \left( \frac{\partial y_2^*}{\partial p_2} \right) - \left( \frac{\partial y_1^*}{\partial p_2} \right) \left( \frac{\partial y_2^*}{\partial p_1} \right) > 0.$$