Exam #2

Do all four problems. Weights: #1 - 25%, #2 - 20%, #3 - 30%, #4 - 25%. Closed book. Open notes. Be sure that your answers are presented in a neat and well-organized manner.

1. Let \( F : \mathbb{R}^n \to \mathbb{R} \) be differentiable and quasi-concave. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be differentiable and quasi-convex. Consider the problem

\[
\max_{\text{w.r.t } x} F(x) \quad \text{subject to} \quad x_1, x_2, \ldots, x_n \geq 0 \quad \text{and} \quad g(x) \leq b,
\]

where \( b \) is a scalar constant. Prove that if \( x^\ast \) is a strict local solution then it is a strict global solution.

2. The following result is known as the Cantor Intersection Theorem.

Let \( F_1, F_2, \ldots, F_n, \ldots \) be an infinite sequence of non-empty, closed, and bounded subsets of \( \mathbb{R}^m \) that are "nested" in the sense that

\[
F_1 \supset F_2 \supset F_3 \supset \ldots \supset F_{n-1} \supset F_n \supset F_{n+1} \supset \ldots.
\]

Then there exists a point \( x \in \mathbb{R}^m \) such that \( x \in F_n \) for all \( n = 1, 2, \ldots \)

Prove the Cantor Intersection Theorem using the Bolzano-Weierstrass Theorem.
(Hint: Form a sequence \( \{x_n\}_{n=1}^\infty \) with \( x_n \in F_n \) for each \( n = 1, 2, \ldots \))

3. Consider the following utility maximization problem:

\[
\max_{\text{w.r.t } x_1, x_2, x_3} U(x_1, x_2, x_3) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I \quad \text{and} \quad x_1, x_2, x_3 \geq 0,
\]

where \( x_1, x_2, \) and \( x_3 \) (\( p_1, p_2, \) and \( p_3 \)) are quantities (positive prices) of goods 1, 2, and 3; \( I > 0 \) is income; and the utility function is given by:

\[
U(x_1, x_2, x_3) = a_1 \ln x_1 + a_2 \ln x_2 + a_3 \ln x_3,
\]

for positive constants \( a_1, a_2, \) and \( a_3 \).
a.) Solve the inequality and non-negativity constrained maximization problem for optimal expenditures on each of the goods.

b.) Now suppose that the consumer's utility maximization problem is subject to an additional rationing constraint, \( x_1 \leq k \) where \( k \) is a positive constant. Assume that parameter values are such that the rationing constraint binds. Solve the problem with the additional rationing constraint for optimal expenditures on each of the goods.

c.) When the rationing constraint binds, the consumer is unable to spend as much on good 1 as she would like. Let's denote this difference between desired and actual expenditures on good 1 by \( E \). Show that \( E \) is reallocated to expenditure on goods 2 and 3 in the following proportions:

\[
\frac{a_2}{a_2 + a_3} E \text{ is spent on good 2 and } \frac{a_3}{a_2 + a_3} E \text{ is spent on good 3.}
\]

4. Consider the problem:

\[
\min_{w.r.t. x} f(x; \theta) \text{ subject to } g(x) = 0,
\]

where \( f(\cdot) \) and \( g(\cdot) \) are differentiable, real-valued functions, \( x \) is an \( n \times 1 \) vector of choice variables, and \( \theta \) is a scalar parameter. Assume that, for each value of \( \theta \), there is a global solution given by a differentiable function of \( \theta \): \( x^*(\theta) \). Define the value function:

\[
f^*(\theta) = f(x^*(\theta); \theta)
\]

a.) State (without proof) the first- and second-order envelope properties for this problem. (They relate the first and second derivatives of \( f^*(\cdot) \) to derivatives of \( f(\cdot) \).)

b.) Use the results stated in part a to prove that the consumer's Hicksian (compensated) demand curves have non-positive slopes.