Homework

(Note: These problems are very closely related to the problems in the required homework set from fall 2006 for which a solution outline is available on the web page.)

1. A firm uses two inputs, \( x_1 \) and \( x_2 \), to produce output via the production function:
   \( y = f(x_1, x_2) \). The firm faces parametric prices for the inputs: \( w_1 \) and \( w_2 \). For given output, \( y \), the cost-minimizing firm's problem is

   \[
   \min_{w, f, x_1, x_2} C(x_1, x_2; w_1, w_2) \quad \text{subject to} \quad f(x_1, x_2) = y,
   \]

   where \( C(x_1, x_2; w_1, w_2) \equiv w_1 x_1 + w_2 x_2 \). Assume that, for each vector of strictly positive factor prices and output, the firm's problem has a regular global solution so that cost-minimizing factor employment levels exist as differentiable functions of prices and output:

   \( x_1^*(y; w_1, w_2) \) and \( x_2^*(y; w_1, w_2) \).

   These are called “conditional factor demand functions,” because they are conditional on a particular output level. Define the value (cost) function:

   \[
   C^*(y; w_1, w_2) \equiv w_1 x_1^*(y; w_1, w_2) + w_2 x_2^*(y; w_1, w_2).
   \]

   Also define the following function:

   \[
   F(y; w_1, w_2; x_1, x_2) \equiv C(x_1, x_2; w_1, w_2) - C^*(y; w_1, w_2).
   \]

   Notice that \( F(\cdot) \geq 0 \) for all values of its arguments that satisfy the constraint:
   \( f(x_1, x_2) - y = 0 \). (For a given \( w_1 \) and \( w_2 \), and for \( x_1, x_2, \) and \( y \) satisfying \( f(x_1, x_2) - y = 0 \), \( C(x_1, x_2; w_1, w_2) \) is one cost of producing \( y \), while \( C^*(y; w_1, w_2) \) is the minimum cost of producing \( y \).) Notice also that, subject to constraint, \( F(\cdot) \); viewed as a function of five choice variables: \( y, w_1, w_2, x_1, \) and \( x_2 \); has a local minimum (of 0) at every point of the form:

   \( (y; w_1, w_2; x_1^*(y; w_1, w_2), x_2^*(y; w_1, w_2)) \).
a.) Write down the first-order necessary conditions for these constrained local minima and show that they provide an alternative proof of the envelope theorem.

b.) Write down the second-order sufficient condition for these constrained local minima and show that it implies the following restrictions on comparative static derivatives:

\[ \frac{\partial x^*_1}{\partial w_1} < 0, \quad \frac{\partial x^*_2}{\partial w_2} < 0, \quad \frac{\partial x^*_1}{\partial w_1} \frac{\partial x^*_2}{\partial w_2} - \left( \frac{\partial x^*_1}{\partial w_2} \right)^2 > 0. \]

(Hint: These implications will probably be easiest to see if you form the bordered Hessian with the arguments ordered as in my notation for \( F(\cdot): F(y; w_1, w_2; x_1, x_2) \).)

2. Consider essentially the same scenario described in problem 1 but this time allow for three factors of production: \( f(x_1, x_2, x_3) = y \). And, to simplify notation, fix the prices of \( x_2 \) and \( x_3 \) at 1 so that the optimization problem is:

\[ \min_{w, y, t, x_1, x_2; x_3 \text{ given } y, w_1} C(x_1, x_2, x_3; w_1) \quad \text{subject to} \quad f(x_1, x_2, x_3) = y, \]

where \( C(x_1, x_2, x_3; w_1) \equiv w_1 x_1 + x_2 + x_3 \), yielding conditional factor demands \( x^*_1(y; w_1), \quad x^*_2(y; w_1), \quad x^*_3(y; w_1) \), and cost function \( C^*(y; w_1) \equiv w_1 x^*_1(y; w_1) + x^*_2(y; w_1) + x^*_3(y; w_1) \).

Inputs 1 and 2 are “variable” and input 3 is “fixed.” The problem described above is the firm’s “long-run” cost minimization problem. It applies to a time frame in which all three inputs can be adjusted and yields long-run conditional factor demands and the long-run cost function. The corresponding “short-run” problem is:

\[ \min_{w, y, t, x_1, x_2; x_3 \text{ given } y, x_1, x_2} C(x_1, x_2, x_3; w_1) \quad \text{subject to} \quad f(x_1, x_2, x_3) = y, \]

yielding the short-run conditional factor demands \( x^*_1(y; w_1; x_3), \quad x^*_2(y; w_1; x_3) \) and the short-run cost function \( C^*(y; w_1, x_3) = w_1 x^*_1(y; w_1, x_3) + x^*_2(y; w_1, x_3) + x_3 \). For given values of \( y \) and \( w_1 \), say \( y^0 \) and \( w_{10} \), we have:

\[ C^*(y^0; w_{10}) = C^*(y^0; w_{10}; x^*_1(y^0; w_{10}), x^*_2(y^0; w_{10})) = C(x^*_1(y^0; w^0_{10}), x^*_2(y^0; w^0_{10}), x^*_3(y^0; w^0_{10}; w_{10})). \]

However, for any \( w_1 \neq w_{10} \), we have:

\[ C^*(y^0; w_{1}) \leq C^*(y^0; w_{1}; x^*_1(y^0; w_{1}), x^*_2(y^0; w_{1})) \leq C(x^*_1(y^0; w^0_{1}), x^*_2(y^0; w^0_{1}), x^*_3(y^0; w^0_{1}; w_{1})). \]

These inequalities follow because the removal of a constraint cannot worsen, and might improve, the optimal value of the objective. The term on the right is the cost of
producing output $y^0$ using factor employment levels that are not necessarily optimal for factor price $w_i$. The term in the middle is cost holding $x_3$ fixed at the same not-necessarily-optimal level, but allowing optimization with respect to $x_1$ and $x_2$. Finally, the term on the left incorporates optimization with respect to $x_1$, $x_2$, and $x_3$. Now imagine graphing each of these three as functions of $w_i$ for a fixed $w_i^0$ and $y^0$. Noting that $C(\cdot)$ is linear in $w_i$ (with slope equal to $x_1^0(y^0;w_i^0)>0$), the graph will look like the one below.

The graph reveals a particular relationship among the second partial derivatives of $C(\cdot)$, $C^*(\cdot)$, and $C^*(\cdot)$ with respect to $w_i$ evaluated at $w_i^0$. Use this relationship to establish the following restrictions on slopes of the long- and short-run conditional factor demands for $x_1$:

For any values of $w_i^0$ and $y^0$: $\frac{\partial x_1^*(y^0;w_i^0)}{\partial w_i} \leq \frac{\partial x_1^*(y^0;w_i^0;x_2^*(y^0;w_i^0);x_3^*(y^0;w_i^0);\cdot)}{\partial w_i^0} \leq 0$.