Econ 600
Schroeter
Fall 2010

## Homework

1. A firm uses two inputs to produce a single output according to the production function: $y = f(x_1, x_2)$, where $y$ denotes the quantity of output and $x_1, x_2$ denote the quantities of the two inputs. Both marginal products, $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ are strictly positive throughout input space. Output is sold, and inputs are purchased, at parametric prices denoted $p$, $w_1$, and $w_2$.

*a*.) Set up the firm's profit maximization problem as an *equality-constrained* maximization problem. Write down the first-order conditions. State the second-order sufficient conditions and derive their implications for the production function.

*b*.) Assuming that the second-order sufficient conditions are satisfied, and using $y^*$ and $x_i^*$ for $i = 1, 2$ to denote profit maximizing output and employment levels, show that

(*i*.) $\frac{\partial y^*}{\partial p} > 0.$

(*ii*.) $\frac{\partial y^*}{\partial w_1} = -\frac{\partial x_1^*}{\partial p}$.

Notes: This problem can also be formulated as an unconstrained optimization problem. But setting it up as a constrained optimization problem makes it somewhat easier to get the results in part *b*. The result in part *b*, *ii* is called a "reciprocity relation." It is also true that $\frac{\partial y^*}{\partial w_2} = -\frac{\partial x_2^*}{\partial p}$.

2. **Note: This problem is very similar to the Homework from Fall 2007. The main goal of this problem is to get you to carefully review the solution outline for the F07 assignment, which is available on the course web page.**

Given differentiable functions:

$$F(x;\alpha):\Re^n x\Re^m \to \Re, \quad g(x;\alpha):\Re^n x\Re^m \to \Re, \quad \text{and} \quad h(x;\alpha):\Re^n x\Re^m \to \Re,$$

consider the following optimization problems:

$$(i) \qquad \min_{w.r.t.\ x,\ given\ \alpha} F(x;\alpha) \quad \text{subject to} \quad g(x;\alpha)=0$$

$$(ii) \qquad \min_{w.r.t.\ x,\ given\ \alpha} F(x;\alpha) \quad \text{subject to} \quad g(x;\alpha)=0 \quad \text{and} \quad h(x;\alpha)=0$$

In these problems, $x \in \Re^n$ (where $n > 2$) is a vector of choice variables and $\alpha \in A$, a subset of $\Re^m$, is a vector of parameters. Assume that each problem has a regular global solution for each $\alpha \in A$ implying that the optimal values of the choice variables and the equilibrium values of the Lagrange multipliers are differentiable functions of $\alpha$:

$$x^*(\alpha),\ \lambda^*(\alpha) \ \text{ for problem } (i) \text{ and } \hat{x}(\alpha),\ \hat{\lambda}(\alpha),\ \hat{\mu}(\alpha) \text{ for problem } (ii).$$

Define the value functions for the two problems:

$$F^*(\alpha) \equiv F(x^*(\alpha);\alpha) \ \text{ and } \ \hat{F}(\alpha) \equiv F(\hat{x}(\alpha);\alpha).$$

For a given value of the parameter vector, $\alpha_0 \in A$, assume that $h(x^*(\alpha_0);\alpha_0)=0$. (That is, the additional constraint in problem (*ii*) is satisfied at problem (*i*)'s solution for $\alpha = \alpha_0$.) Further assume that:

$$\frac{\partial g}{\partial x}(x^*(\alpha_0);\alpha_0) \ \text{ and } \ \frac{\partial h}{\partial x}(x^*(\alpha_0);\alpha_0) \ \text{ are linearly independent.}$$

*a*.) As in the F07 Homework, it is obvious that $\hat{x}(\alpha_0) = x^*(\alpha_0)$ and $\hat{F}(\alpha_0) = F^*(\alpha_0)$. Review the F07 Homework's proof that

$$\frac{\partial F^*}{\partial \alpha}(\alpha_0) = \frac{\partial \hat{F}}{\partial \alpha}(\alpha_0)$$

and convince yourself that the proof goes through in this case as well.

*b*.) Review the F07 Homework's proof of a second order relation involving value function derivatives and convince yourself that that proof can be adapted to show, for this case:

$$\frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0) - \frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0) \text{ is } \textit{positive} \text{ semi-definite.} \qquad (*)$$

(Hint: You don't actually have to write anything down for parts *a* and *b*. I just want you to study the solution outline for the F07 Homework and to understand it.)

*c*.) A firm produces a single output using three inputs: Two inputs are "variable" (in both the short- and long-run) and one, called "plant," is "fixed" (in the short-run but variable in the long-run). Use the framework of this problem to prove the following proposition:

> At a given output level, long-run marginal cost is no steeper than short-run marginal cost in the plant that is the optimal size for that output level.

To do this, you'll have to specify the nature of the $x$ and $\alpha$ vectors, the forms of the $F(\cdot)$, $g(\cdot)$, and $h(\cdot)$ functions, and the restriction on the $\alpha_0$ vector to insure that $h(x^*(\alpha_0);\alpha_0) = 0$. Then you'll have to explain how (*) can be used to prove the proposition.