Econ 600
Schroeter
Fall 2010

Homework solution outline

1. *a*.) $\max_{w.r.t.\ y,\, x_1,\, x_2} py - w_1 x_1 - w_2 x_2$ subject to $f(x_1, x_2) - y = 0$

Lagrangian: $L(y, x_1, x_2; \lambda) = py - w_1 x_1 - w_2 x_2 + \lambda(y - f(x_1, x_2))$

FONC:

$$\frac{\partial L}{\partial \lambda}(*) = y^* - f(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial y}(*) = p + \lambda^* = 0$$

$$\frac{\partial L}{\partial x_1}(*) = -w_1 - \lambda^* f_1(x_1^*, x_2^*) = 0$$

$$\frac{\partial L}{\partial x_2}(*) = -w_2 - \lambda^* f_2(x_1^*, x_2^*) = 0$$

. . . where subscripts on $f(\cdot)$ denote partial derivatives.

The SOSC are in terms of the bordered Hessian:

$$\overline{H} = \begin{bmatrix} 0 & -1 & f_1 & f_2 \\ -1 & 0 & 0 & 0 \\ f_1 & 0 & -\lambda f_{11} & -\lambda f_{12} \\ f_2 & 0 & -\lambda f_{12} & -\lambda f_{22} \end{bmatrix}\Bigg|_{(y,\, x_1,\, x_2,\, \lambda) = (y^*,\, x_1^*,\, x_2^*,\, \lambda^*)}$$

In the notation of lecture, the SOSC are $\det \overline{H}_2 > 0$ and $\det \overline{H}_3 = \det \overline{H} < 0$.

$$\det \overline{H}_2 = \det \begin{bmatrix} 0 & -1 & f_1 \\ -1 & 0 & 0 \\ f_1 & 0 & -\lambda^* f_{11} \end{bmatrix} = \lambda^* f_{11} > 0 \Rightarrow f_{11} < 0 \quad \text{(because } \lambda^* < 0 \text{ from FONC)}$$

$$\det \overline{H} = -\lambda^{*2}\left(f_{11} f_{22} - f_{12}^2\right) < 0 \Rightarrow f_{11} f_{22} - f_{12}^2 > 0 \Rightarrow f_{22} < 0 \quad \text{(because } f_{11} < 0\text{)}$$

So the Hessian of $f(\cdot)$ has leading principal minors that alternate in sign beginning with negative. This means that the Hessian of $f(\cdot)$ is negative definite at the optimum. The Jacobian of the system of FONCs is $\overline{H}$ with factors of -1 in the first row and column, and therefore has the same determinant as $\overline{H}$. Thus, the SOSC insures that the key hypothesis of the implicit function theorem holds, and therefore that the FONC implicitly define λ^*, y^*, x_1^*, and x_2^* as differentiable functions of the parameters.

b.) (*i*.) Differentiating the FONC with respect to *p*:

$$\begin{bmatrix} 0 & 1 & -f_1 & -f_2 \\ 1 & 0 & 0 & 0 \\ -f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\ -f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22} \end{bmatrix} \begin{pmatrix} \frac{\partial \lambda^*}{\partial p} \\ \frac{\partial y^*}{\partial p} \\ \frac{\partial x_1^*}{\partial p} \\ \frac{\partial x_2^*}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Solving by Cramer's Rule:

$$\frac{\partial y^*}{\partial p} = \frac{1}{\det \overline{H}} \cdot \det \begin{bmatrix} 0 & 0 & -f_1 & -f_2 \\ 1 & -1 & 0 & 0 \\ -f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\ -f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22} \end{bmatrix} = \frac{-\lambda^*}{\det \overline{H}} \left(f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2 \right).$$

The term in parentheses is of the form $z' \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} z$, where $z' = (-f_2,\ f_1)$. As a quadratic form in a negative definite matrix, this term is negative. Since $\det \overline{H}$ and λ^* are both also negative, we have the result:

$$\frac{\partial y^*}{\partial p} > 0 .$$

(*ii*.) Solving the system for $\partial x_1^* / \partial p$ by Cramer's Rule:

$$\frac{\partial x_1^*}{\partial p} = \frac{1}{\det \overline{H}} \cdot \det \begin{bmatrix} 0 & 1 & 0 & -f_2 \\ 1 & 0 & -1 & 0 \\ -f_1 & 0 & 0 & -\lambda^* f_{12} \\ -f_2 & 0 & 0 & -\lambda^* f_{22} \end{bmatrix} = \frac{-\lambda^*}{\det \overline{H}} \left(f_1 f_{22} - f_2 f_{12} \right)$$

Differentiating the FONC with respect to w_1:

$$\begin{bmatrix} 0 & 1 & -f_1 & -f_2 \\ 1 & 0 & 0 & 0 \\ -f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\ -f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22} \end{bmatrix} \begin{pmatrix} \frac{\partial \lambda^*}{\partial w_1} \\ \frac{\partial y^*}{\partial w_1} \\ \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Solving by Cramer's Rule:

$$\frac{\partial y^*}{\partial w_1} = \frac{1}{\det \overline{\overline{H}}} \cdot \det \begin{bmatrix} 0 & 0 & -f_1 & -f_2 \\ 1 & 0 & 0 & 0 \\ -f_1 & 1 & -\lambda^* f_{11} & -\lambda^* f_{12} \\ -f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22} \end{bmatrix} = \frac{\lambda^*}{\det \overline{\overline{H}}} \left(f_1 f_{22} - f_2 f_{12} \right) = -\frac{\partial x_1^*}{\partial p}.$$

2. Let x_1, x_2, and x_3 denote employment levels of the variable factors (indexed by 1 and 2) and the fixed factor (3), respectively. Likewise, let w_1, w_2, and w_3 denote the corresponding opportunity costs of the factors. Let *y* denote the output level and let $\bar{x}_3$ denote a specific "plant size." Take the vector of choice variables to be $x' = (x_1, x_2, x_3)$ and the vector of parameters to be $\alpha' = (w_1, w_2, w_3, y, \bar{x}_3)$. The objective function is

$$f(x; \alpha) \equiv w_1 x_1 + w_2 x_2 + w_3 x_3$$

And the constraint functions are

$$g(x; \alpha) \equiv f(x_1, x_2, x_3) - y \quad \text{and}$$

$$h(x; \alpha) \equiv x_3 - \bar{x}_3.$$

Problem *i* is long-run cost minimization:

$$\min_{w.r.t.\, x_1, x_2, x_3} w_1 x_1 + w_2 x_2 + w_3 x_3 \quad \text{subject to} \quad f(x_1, x_2, x_3) - y = 0$$

with Lagrangian: $L_i = w_1 x_1 + w_2 x_2 + w_3 x_3 + \lambda (y - f(x_1, x_2, x_3))$ and solutions:

$x_1^*(w_1, w_2, w_3, y)$, $x_2^*(w_1, w_2, w_3, y)$, $x_3^*(w_1, w_2, w_3, y)$, and $\lambda^*(w_1, w_2, w_3, y)$.

The value function for this problem, $F^*(w_1, w_2, w_3, y)$, is the long-run cost function.

Problem *ii* is short-run cost minimization in plant size $x_3 = \bar{x}_3$:

$$\min_{w.r.t.\ x_1,\, x_2,\, x_3} w_1 x_1 + w_2 x_2 + w_3 x_3 \ \text{ subject to } \ f(x_1, x_2, x_3) - y = 0 \ \text{ and } \ x_3 - \bar{x}_3 = 0$$

with Lagrangian: $L_{ii} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \lambda(y - f(x_1, x_2, x_3)) + \mu(\bar{x}_3 - x_3)$ and solutions:

$\hat{x}_1(w_1, w_2, w_3, y, \bar{x}_3)$, $\hat{x}_2(w_1, w_2, w_3, y, \bar{x}_3)$, $\hat{\lambda}(w_1, w_2, w_3, y, \bar{x}_3)$,

and $\hat{\mu}(w_1, w_2, w_3, y, \bar{x}_3)$.

The value function for this problem, $\hat{F}(w_1, w_2, w_3, y, \bar{x}_3)$, is the short-run cost function for plant size $\bar{x}_3$.

Consider specific values for the parameters: $\alpha_0' = (w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30})$. In order for the second constraint to be satisfied at problem *i*'s solution for $\alpha = \alpha_0$, we need: $\bar{x}_{30} = x_3^*(w_{10}, w_{20}, w_{30}, y_0)$. In other words, $\bar{x}_{30}$ must be the long-run-optimal plant size for output y_0 at factor prices w_{10}, w_{20}, and w_{30}. The fact that $\frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0) - \frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0)$ is positive semi-definite implies that diagonal elements must be non-negative. Thus:

$$\frac{\partial^2 \hat{F}}{\partial y^2}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) - \frac{\partial^2 F^*}{\partial y^2}(w_{10}, w_{20}, w_{30}, y_0) \geq 0.$$

By the envelope theorem, this implies:

$$\frac{\partial}{\partial y}\hat{\lambda}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) - \frac{\partial}{\partial y}\lambda^*(w_{10}, w_{20}, w_{30}, y_0) \geq 0,$$

where $\lambda^*(w_{10}, w_{20}, w_{30}, y_0)$ and $\hat{\lambda}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30})$ are, respectively, long-run marginal cost at y_0 and short-run marginal cost at y_0 in the plant ($\bar{x}_{30}$) that is optimal for output y_0. Therefore the inequality above verifies the claim: Long-run marginal cost is no steeper than short-run marginal cost.