Homework solution outline

1. a.) \[ \max_{w.r.t. \ y, x_1, x_2} \ p y - w_1 x_1 - w_2 x_2 \ \text{subject to} \ f(x_1, x_2) - y = 0 \]

Lagrangian: \[ L(y, x_1, x_2; \lambda) = p y - w_1 x_1 - w_2 x_2 + \lambda (y - f(x_1, x_2)) \]

FONC:
\[ \frac{\partial L}{\partial \lambda} (\ast) = y^* - f(x_1^*, x_2^*) = 0 \]
\[ \frac{\partial L}{\partial y} (\ast) = p + \lambda^* = 0 \]
\[ \frac{\partial L}{\partial x_1} (\ast) = -w_1 - \lambda^* f_1(x_1^*, x_2^*) = 0 \]
\[ \frac{\partial L}{\partial x_2} (\ast) = -w_2 - \lambda^* f_2(x_1^*, x_2^*) = 0 \]

\[ \ldots \text{where subscripts on } f(\cdot) \text{ denote partial derivatives.} \]

The SOSC are in terms of the bordered Hessian:
\[ \overline{H} = \begin{bmatrix}
0 & -1 & f_1 & f_2 \\
-1 & 0 & 0 & 0 \\
f_1 & 0 & -\lambda f_{11} & -\lambda f_{12} \\
f_2 & 0 & -\lambda f_{12} & -\lambda f_{22}
\end{bmatrix}_{(y, x_1, x_2, \lambda) = (y^*, x_1^*, x_2^*, \lambda^*)} \]

In the notation of lecture, the SOSC are \[ \det \overline{H}_2 > 0 \quad \text{and} \quad \det \overline{H}_3 = \det \overline{H} < 0. \]

\[ \det \overline{H}_2 = \det \begin{bmatrix}
0 & -1 & f_1 \\
-1 & 0 & 0 \\
f_1 & 0 & -\lambda f_{11}
\end{bmatrix} = \lambda f_{11} > 0 \Rightarrow f_{11} < 0 \quad \text{(because } \lambda^* < 0 \text{ from FONC)} \]

\[ \det \overline{H} = -\lambda^2 (f_{11} f_{22} - f_{12}^2) < 0 \Rightarrow f_{11} f_{22} - f_{12}^2 > 0 \Rightarrow f_{22} < 0 \quad \text{(because } f_{11} < 0) \]
So the Hessian of \( f(\cdot) \) has leading principal minors that alternate in sign beginning with negative. This means that the Hessian of \( f(\cdot) \) is negative definite at the optimum. The Jacobian of the system of FONCs is \( \bar{H} \) with factors of -1 in the first row and column, and therefore has the same determinant as \( \bar{H} \). Thus, the SOSC insures that the key hypothesis of the implicit function theorem holds, and therefore that the FONC implicitly define \( \lambda^*, y^*, x_1^*, \) and \( x_2^* \) as differentiable functions of the parameters.

\( b. \) (i.) Differentiating the FONC with respect to \( p \):

\[
\frac{\partial \lambda^*}{\partial p} = \frac{1}{\det H} \cdot \text{det} \begin{bmatrix}
0 & 0 & -f_1 & -f_2 \\
1 & -1 & 0 & 0 \\
-f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\
-f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \lambda^*}{\partial p} \\
\frac{\partial y^*}{\partial p} \\
\frac{\partial x_1^*}{\partial p} \\
\frac{\partial x_2^*}{\partial p}
\end{bmatrix} = \begin{bmatrix}
0 \\
-1 \\
0 \\
0
\end{bmatrix}.
\]

Solving by Cramer’s Rule:

\[
\frac{\partial y^*}{\partial p} = \frac{1}{\det H} \cdot \text{det} \begin{bmatrix}
0 & 0 & -f_1 & -f_2 \\
1 & -1 & 0 & 0 \\
-f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\
-f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22}
\end{bmatrix} \begin{bmatrix}
0 \\
-1 \\
0 \\
0
\end{bmatrix} = -\lambda^* \frac{f_{11} f_2^2 - 2 f_{12} f_1 f_2 + f_{22} f_1^2}{\det H}.
\]

The term in parentheses is of the form \( z' \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} z \), where \( z' = (-f_2, f_1) \). As a quadratic form in a negative definite matrix, this term is negative. Since \( \det \bar{H} \) and \( \lambda^* \) are both also negative, we have the result:

\[
\frac{\partial y^*}{\partial p} > 0.
\]

(ii.) Solving the system for \( \frac{\partial x_i^*}{\partial p} \) by Cramer’s Rule:

\[
\frac{\partial x_i^*}{\partial p} = \frac{1}{\det H} \cdot \text{det} \begin{bmatrix}
0 & 1 & 0 & -f_2 \\
1 & 0 & -1 & 0 \\
-f_1 & 0 & 0 & -\lambda^* f_{12} \\
-f_2 & 0 & 0 & -\lambda^* f_{22}
\end{bmatrix} = -\lambda^* \frac{f_1 f_{22} - f_2 f_{12}}{\det H}.
\]
Differentiating the FONC with respect to $w_1$:

$$
\begin{bmatrix}
0 & 1 & -f_1 & -f_2 \\
1 & 0 & 0 & 0 \\
-f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\
-f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \lambda^*}{\partial w_1} \\
\frac{\partial y^*}{\partial w_1} \\
\frac{\partial y^*}{\partial x_1} \\
\frac{\partial y^*}{\partial x_2}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
$$

Solving by Cramer’s Rule:

$$
\frac{\partial y^*}{\partial w_1} = \frac{1}{\det H} \cdot \text{det}
\begin{bmatrix}
0 & 0 & -f_1 & -f_2 \\
1 & 0 & 0 & 0 \\
-f_1 & 0 & -\lambda^* f_{11} & -\lambda^* f_{12} \\
-f_2 & 0 & -\lambda^* f_{12} & -\lambda^* f_{22}
\end{bmatrix}
= \frac{\lambda^*}{\det H} (f_1 f_{22} - f_{2} f_{12}) = -\frac{\partial x^*_1}{\partial p}.
$$

2. Let $x_1$, $x_2$, and $x_3$ denote employment levels of the variable factors (indexed by 1 and 2) and the fixed factor (3), respectively. Likewise, let $w_1$, $w_2$, and $w_3$ denote the corresponding opportunity costs of the factors. Let $y$ denote the output level and let $\bar{x}_3$ denote a specific “plant size.” Take the vector of choice variables to be $\mathbf{x} = (x_1, x_2, x_3)$ and the vector of parameters to be $\alpha' = (w_1, w_2, w_3, y, \bar{x}_3)$. The objective function is

$$
f(x; \alpha) \equiv w_1 x_1 + w_2 x_2 + w_3 x_3
$$

And the constraint functions are

$$
g(x; \alpha) \equiv f(x_1, x_2, x_3) - y \quad \text{and}
$$

$$
h(x; \alpha) \equiv x_3 - \bar{x}_3.
$$

Problem $i$ is long-run cost minimization:

$$
\min_{w.r.t. x_1, x_2, x_3} w_1 x_1 + w_2 x_2 + w_3 x_3 \quad \text{subject to} \quad f(x_1, x_2, x_3) - y = 0
$$

with Lagrangian: $L_i = w_1 x_1 + w_2 x_2 + w_3 x_3 + \lambda(y - f(x_1, x_2, x_3))$ and solutions:
\begin{align*}
x_1^*(w_1, w_2, w_3, y), \quad x_2^*(w_1, w_2, w_3, y), \quad x_3^*(w_1, w_2, w_3, y), \quad \text{and} \quad \lambda^*(w_1, w_2, w_3, y).
\end{align*}

The value function for this problem, \( F^*(w_1, w_2, w_3, y) \), is the long-run cost function.

Problem \( ii \) is short-run cost minimization in plant size \( x_3 = \bar{x}_3 \):

\[
\min_{w.t.t. \, x_1, x_2, x_3} w_1 x_1 + w_2 x_2 + w_3 x_3 \quad \text{subject to} \quad f(x_1, x_2, x_3) - y = 0 \quad \text{and} \quad x_3 - \bar{x}_3 = 0
\]

with Lagrangian: \( L_{ii} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \lambda(y - f(x_1, x_2, x_3)) + \mu(\bar{x}_3 - x_3) \) and solutions:

\[
\hat{x}_1(w_1, w_2, w_3, y, \bar{x}_3), \quad \hat{x}_2(w_1, w_2, w_3, y, \bar{x}_3), \quad \hat{\lambda}(w_1, w_2, w_3, y, \bar{x}_3),
\]

and \( \hat{\mu}(w_1, w_2, w_3, y, \bar{x}_3) \).

The value function for this problem, \( \hat{F}(w_1, w_2, w_3, y, \bar{x}_3) \), is the short-run cost function for plant size \( \bar{x}_3 \).

Consider specific values for the parameters: \( \alpha' = (w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) \). In order for the second constraint to be satisfied at problem \( i \)'s solution for \( \alpha = \alpha_0 \), we need:

\( \bar{x}_{30} = x_3^*(w_{10}, w_{20}, w_{30}, y_0) \). In other words, \( \bar{x}_{30} \) must be the long-run-optimal plant size for output \( y_0 \) at factor prices \( w_{10}, w_{20}, \) and \( w_{30} \). The fact that \( \frac{\partial^2 \hat{F}(\alpha_0)}{\partial \alpha^2} - \frac{\partial^2 F^*(\alpha_0)}{\partial \alpha^2} \) is positive semi-definite implies that diagonal elements must be non-negative. Thus:

\[
\frac{\partial^2 \hat{F}}{\partial y^2}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) - \frac{\partial^2 F^*}{\partial y^2}(w_{10}, w_{20}, w_{30}, y_0) \geq 0.
\]

By the envelope theorem, this implies:

\[
\frac{\partial}{\partial y} \hat{\lambda}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) - \frac{\partial}{\partial y} \lambda^*(w_{10}, w_{20}, w_{30}, y_0) \geq 0,
\]

where \( \lambda^*(w_{10}, w_{20}, w_{30}, y_0) \) and \( \hat{\lambda}(w_{10}, w_{20}, w_{30}, y_0, \bar{x}_{30}) \) are, respectively, long-run marginal cost at \( y_0 \) and short-run marginal cost at \( y_0 \) in the plant \( \bar{x}_{30} \) that is optimal for output \( y_0 \). Therefore the inequality above verifies the claim: Long-run marginal cost is no steeper than short-run marginal cost.