1. Let $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be differentiable functions of a vector of arguments $(x,z,\theta)$ where $x$ and $z$ are vectors of choice variables of dimensions $n \geq 1$ and $m \geq 1$, respectively, and $\theta$ is a scalar parameter. Consider the problem:

$$ \min_{w.r.t.\ x, z} F(x,z,\theta) \text{ subject to } g(x,z,\theta) = 0. $$

Assume that the problem has a strict global solution for each value of $\theta$. Denote the optimal values of $x$ and $z$ by $x^*(\theta)$ and $z^*(\theta)$ (assumed differentiable) and define the value function:

$$ F^*(\theta) \equiv F(x^*(\theta),z^*(\theta),\theta). $$

Now consider the problem:

$$ \min_{w.r.t.\ x} F(x,z,\theta) \text{ subject to } g(x,z,\theta) = 0. $$

Assume that the problem has a strict global solution for each vector of values of $z$ and $\theta$. Denote the optimal value of $x$ by $\hat{x}(z,\theta)$ (assumed differentiable) and define the value function:

$$ F^*(\theta) \equiv F(\hat{x}(z,\theta),z,\theta). $$

Problem #3 from Homework #2 in Fall 2000 established the following first- and second-order envelope properties for these value functions:

$$ \frac{dF^*}{d\theta}(\theta) = \frac{\partial \hat{F}}{\partial \theta}(z^*(\theta),\theta) \quad \text{and} \quad \frac{d^2F^*}{d\theta^2}(\theta) \leq \frac{\partial^2 \hat{F}}{\partial \theta^2}(z^*(\theta),\theta). $$

Verify these properties by direct calculation for the case in which $n = m = 1$, the role of "$F$" is played by $wL + rK$, and the role of "$g$" is played by $L^\alpha K^\beta - Q$, where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. The interpretations are as follows:

$L$ (the "$x$" variable) is labor employment,
$K$ (the "$z$" variable) is capital employment,
$w$ and $r$ are the unit opportunity costs of labor and capital, respectively, and
$Q$ (the "$\theta$" parameter) is output quantity given by production function $Q = L^\alpha K^\beta$.

With these interpretations, $F^*(\cdot)$ and $\hat{F}(\cdot)$ are the long- and short-run cost functions, respectively.
2. \( F : \mathbb{R}^n \to \mathbb{R} \) is a differentiable function, \( g \in \mathbb{R}^n \) is a 1 x \( n \) vector with \( g \neq 0 \), and \( b \) is a scalar constant. Consider the problem:

\[
\max_{x} F(x) \quad \text{subject to} \quad g \cdot x = b, \quad (*)
\]

and define the Lagrangian:

\[
L(x; \lambda) = F(x) + \lambda(b - g \cdot x).
\]

In lecture we sketched the graph and intuition supporting the following proposition:

If \( F(\cdot) \) is quasi-concave and \( x^* \) satisfies the first-order necessary conditions (that is, there exists \( \lambda^* \neq 0 \) such that \( (x^*, \lambda^*) \) is a stationary point of \( L(\cdot) \)), then \( x^* \) is a global solution to problem (*).

In this problem, we will provide a formal proof of this proposition.

Here are some hints. First recall that quasi-concavity of \( F(\cdot) \) is equivalent to:

\[
\text{For all } u, v \in \mathbb{R}^n \text{ such that } F(u) \geq F(v), \quad \frac{\partial F}{\partial x}(v)(u - v) \geq 0.
\]

Next, recalling a result from math camp about the dot product of two vectors (Simon and Blume, Theorem 10.3), we can see that the last inequality above has a geometric interpretation: The vector between points \( u \) and \( v \), and the gradient vector of \( F(\cdot) \) at \( v \), form an angle that is no bigger than 90 degrees.

Refer to the graph on the following page. Let \( x^* \) satisfy the first order conditions. To construct a proof by contradiction, suppose that there is an \( \hat{x} \) that satisfies the constraint but is such that \( F(\hat{x}) > F(x^*) \). By continuity of \( F(\cdot) \), the value of the function must be greater than \( F(x^*) \) on some neighborhood about \( \hat{x} \). Thus, one can add to \( \hat{x} \) a small positive multiple \( (\varepsilon > 0) \) of the vector \(-\partial F/\partial x(x^*)\) and still have a point at which the value of the objective function is greater than \( F(x^*) \). Use this to establish a contradiction with the quasi-concavity of \( F(\cdot) \).
\[ \frac{\partial F}{\partial x} (x^*) \]

\[ F(x) = \text{constant} \]

\[ g \cdot x = b \]