

Homework #1

1. For $A \subset \mathfrak{R}$, define the set of lower bounds for A , $L(A)$, as follows:

$$L(A) = \{l \in \mathfrak{R} : l \leq a \text{ for all } a \in A\}.$$

If $L(A) \neq \emptyset$ (A has at least one lower bound; that is, A is *bounded below*), define the infimum of A (or the greatest lower bound of A), denoted $\inf A$, as follows:

$$\inf A = l^* \text{ such that } l^* \in L(A) \text{ and } l^* \geq l \text{ for all } l \in L(A).$$

It is a fundamental property of the real numbers that any set of real numbers that is bounded below has an infimum and it is well-defined by the statement above. In other words, for any A , bounded below, there is a unique l^* that satisfies the stated conditions.

Let $A = \{a_i : i \in I\}$, $B = \{b_i : i \in I\}$, $C = \{c_i : i \in I\}$ and $D = \{d_i : i \in I\}$ be finite or infinite sets of real numbers indexed by the same set, I . (I might be the set of positive integers, for example, in which case, A , B , C , and D would each contain countably infinitely many elements.)

a.) Suppose that both A and B are bounded below and that $a_i \geq b_i$ for all $i \in I$. Prove

$$\inf A \geq \inf B.$$

b.) Suppose that both C and D are bounded below and let α and β be positive real numbers. Prove

$$\inf \{\alpha c_i + \beta d_i : i \in I\} \geq \alpha \inf C + \beta \inf D.$$

c.) Let $f_i : \mathfrak{R}^n \rightarrow \mathfrak{R}$ for $i \in I$, be a finite or infinite collection of concave functions. Define $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ as follows:

$$\text{For } x \in \mathfrak{R}^n, f(x) = \inf \{f_i(x) : i \in I\}.$$

(Assume that the set that appears on the right-hand-side of this equation is bounded below for all $x \in \mathfrak{R}^n$.) Use the results of a and b to prove that $f(\cdot)$ is concave.

2. Let $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$ be a concave function. Let A be an $n \times m$ matrix and let $b \in \mathfrak{R}^n$. Consider the function $g : \mathfrak{R}^m \rightarrow \mathfrak{R}$ defined by:

$$\text{For } x \in \mathfrak{R}^m, g(x) = f(Ax + b).$$

Prove that $g(\cdot)$ is concave.

3. In lecture we showed that $f : \mathfrak{R}_+^2 \rightarrow \mathfrak{R}$ defined by the formula $f(x_1, x_2) = x_1 \cdot x_2$ is quasi-concave. Use the same formula to define a function on domain \mathfrak{R}^2 instead of \mathfrak{R}_+^2 . Is that function also quasi-concave? Explain.

4. Let U be a convex subset of \mathfrak{R}^n and suppose that the function $f : U \rightarrow \mathfrak{R}$ is both concave and strictly quasi-concave. Is $f(\cdot)$ strictly concave? Either prove or provide a counterexample.

5. Given U , a convex subset of \mathfrak{R}^n we say that a function $F : U \rightarrow \mathfrak{R}$ is quasi-convex if, for all $u, v \in U$, and for all $h \in [0, 1]$,

$$F(hu + (1-h)v) \leq \max\{F(u), F(v)\}.$$

Prove that $F(\cdot)$ is quasi-convex if and only if $-F(\cdot)$ is quasi-concave.