Homework #1

1. For $A \subseteq \mathbb{R}$, define the set of lower bounds for $A$, $L(A)$, as follows:

$$L(A) = \{ l \in \mathbb{R} : l \leq a \text{ for all } a \in A \}.$$ 

If $L(A) \neq \emptyset$ ($A$ has at least one lower bound; that is, $A$ is bounded below), define the infimum of $A$ (or the greatest lower bound of $A$), denoted $\inf A$, as follows:

$$\inf A = l^* \text{ such that } l^* \in L(A) \text{ and } l^* \geq l \text{ for all } l \in L(A).$$

It is a fundamental property of the real numbers that any set of real numbers that is bounded below has an infimum and it is well-defined by the statement above. In other words, for any $A$, bounded below, there is a unique $l^*$ that satisfies the stated conditions.

Let $A = \{ a_i : i \in I \}$, $B = \{ b_i : i \in I \}$, $C = \{ c_i : i \in I \}$ and $D = \{ d_i : i \in I \}$ be finite or infinite sets of real numbers indexed by the same set, $I$. ($I$ might be the set of positive integers, for example, in which case, $A$, $B$, $C$, and $D$ would each contain countably infinitely many elements.)

a.) Suppose that both $A$ and $B$ are bounded below and that $a_i \geq b_i$ for all $i \in I$. Prove

$$\inf A \geq \inf B.$$ 

b.) Suppose that both $C$ and $D$ are bounded below and let $\alpha$ and $\beta$ be positive real numbers. Prove

$$\inf \{ \alpha c_i + \beta d_i : i \in I \} \geq \alpha \inf C + \beta \inf D.$$ 

c.) Let $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in I$, be a finite or infinite collection of concave functions. Define $f : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\text{For } x \in \mathbb{R}^n, f(x) = \inf \{ f_i(x) : i \in I \}.$$ 

(Assume that the set that appears on the right-hand-side of this equation is bounded below for all $x \in \mathbb{R}^n$.) Use the results of $a$ and $b$ to prove that $f(\cdot)$ is concave.
2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a concave function. Let $A$ be an $n \times m$ matrix and let $b \in \mathbb{R}^n$. Consider the function $g : \mathbb{R}^m \to \mathbb{R}$ defined by:

For $x \in \mathbb{R}^m$, $g(x) = f(Ax + b)$.

Prove that $g(\cdot)$ is concave.

3. In lecture we showed that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by the formula $f(x_1, x_2) = x_1 \cdot x_2$ is quasi-concave. Use the same formula to define a function on domain $\mathbb{R}^2$ instead of $\mathbb{R}^+$. Is that function also quasi-concave? Explain.

4. Let $U$ be a convex subset of $\mathbb{R}^n$ and suppose that the function $f : U \to \mathbb{R}$ is both concave and strictly quasi-concave. Is $f(\cdot)$ strictly concave? Either prove or provide a counterexample.

5. Given $U$, a convex subset of $\mathbb{R}^n$ we say that a function $F : U \to \mathbb{R}$ is quasi-convex if, for all $u, v \in U$, and for all $h \in [0, 1]$,

$$F(hu + (1 - h)v) \leq \max\{F(u), F(v)\}.$$  

Prove that $F(\cdot)$ is quasi-convex if and only if $-F(\cdot)$ is quasi-concave.