1. \( f : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable and has a local maximum at \( x^\ast \). In addition,

\[
\frac{d}{dx} f(x) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, n - 1; \quad \text{and} \quad \frac{d^n}{dx^n} f(x^\ast) \neq 0.
\]

(In other words, all of the derivatives of \( f(x) \) from the 1st through the \( n - 1 \)st are zero when evaluated at \( x^\ast \). The \( n^{th} \) derivative is nonzero at \( x^\ast \).) Prove that \( n \) is even and that \( \frac{d^n}{dx^n} f(x^\ast) < 0 \). (Hint: Use Taylor's theorem.)

2. (Simon and Blume 21.13; a proof of Theorem 21.7) Suppose that \( U \) is a convex subset of \( \mathbb{R}^n \), that \( F : U \rightarrow \mathbb{R} \) is differentiable and concave, and that there exists \( x^\ast \in U \) such that

\[
\frac{\partial F}{\partial x}(x^\ast) \cdot (x - x^\ast) \leq 0 \quad \text{for all} \quad x \in U.
\]

Prove that \( x^\ast \) is a global maximizer of \( F(x) \) on \( U \); that is, prove that \( F(x^\ast) \geq F(x) \) for all \( x \in U \). For the case \( n = 1 \) (so that \( U \) is an interval of the real line), draw a graph to illustrate the fact that a global maximizer of \( F(x) \) that occurs at the boundary of \( U \) will satisfy the above derivative property.

3. Read section 22.1 of Simon and Blume. Then consider the utility function

\[
U(x_1, x_2) = x_1^a x_2^b \quad \text{where} \quad a > 0, \ b > 0, \ \text{and} \ \ a + b = 1.
\]

Set up and solve the utility maximization problem to derive the Marshallian demand functions and the indirect utility function. Then set up and solve the expenditure minimization problem to derive the compensated (or Hicksian) demand functions. Use the functional forms you derived to directly verify the Slutsky equation (Simon and Blume Theorem 22.8).

4. Consider the following optimization problem:

\[
\max \ \text{or} \ \min \ x^2 + y^2 + z^2 \quad \text{subject to} \quad ax^2 + by^2 + cz^2 = 1,
\]

where \( a, b, \) and \( c \) are constants satisfying \( a > b > c > 0 \). Write down the Lagrangian and the first-order conditions for this problem. Find all solutions to the first-order conditions.
Check the second-order necessary and sufficient conditions, for both local maxima and minima, at each of the solutions to the first-order conditions.

(Note: For some geometric intuition to accompany this problem, note that the constraint is the equation of a surface in \( \mathbb{R}^3 \). "Slices" cut through the surface on planes that are perpendicular to any one of the coordinate axes expose elliptical cross-sections. Furthermore, optimizing the objective function for this problem is equivalent to optimizing \( \sqrt{x^2 + y^2 + z^2} \), the Euclidean distance from the origin in \( \mathbb{R}^3 \) to the point \((x, y, z)\). So the problem is to find the points on the constraint surface that are closest to and farthest away from the origin.)