1. Consider the constrained maximization problem:

\[
\max_{\text{w.r.t. } x_1, x_2} F(x_1, x_2) \quad \text{subject to} \quad g(x_1, x_2) = c,
\]

where \( F(\cdot) \) and \( g(\cdot) \) are differentiable, real-valued functions, and \( c \) is a constant. We showed in lecture that the first order conditions for this problem imply a tangency between the constraint curve and the objective function's contour line through the optimum, \((x_1^*, x_2^*)\). The two graphs on the attached page each depict a point satisfying this tangency property. Obviously, \((x_1^*, x_2^*)\) in the first graph is a solution to the problem but \((x_1^*, x_2^*)\) in the second graph is not. The difference is that, in the first graph, the objective function's contour line is "more convex" than the constraint curve at \((x_1^*, x_2^*)\) whereas, in the second graph, the opposite is true. Assuming that the first order partial derivatives of \( F(\cdot) \) are both positive and that the partial of \( g(\cdot) \) with respect to \( x_2 \) is nonzero at \((x_1^*, x_2^*)\) (these assumptions are consistent with the way my graphs are drawn), show that the sufficient conditions for the problem imply that the contour line is "more convex" than the constraint curve at the optimum.

(Hint: To say that the contour line is "more convex" than the constraint curve means \( \phi''(x_1^*) > \phi''(x_1^*) \), where \( \phi(\cdot) \) is implicitly defined by \( F(x_1, \phi(x_1)) = F(x_1^*, x_2^*) \) and \( \phi(\cdot) \) is implicitly defined by \( g(x_1, \phi(x_1)) = c \).

2. In lecture, we proved a theorem that gave first- and second-derivative characterizations of differentiable concave functions. A similar result holds for quasi-concave functions:

**Theorem:** Let \( A \subset \mathbb{R}^n \) be a convex set. If \( F : A \rightarrow \mathbb{R} \) is differentiable, then the following are equivalent:

\[ i. \quad F(hu + (1-h)v) \geq \min\{F(u), F(v)\} \quad \text{for all } u, v \in A \quad \text{and for all } h \in [0, 1]. \]

(That is, \( F(\cdot) \) is quasi-concave.)

\[ ii. \quad \frac{\partial F}{\partial x}(v) \cdot (u - v) \geq 0 \quad \text{for all } u, v \in A \quad \text{such that } F(u) \geq F(v). \]

\[ iii. \quad z' \cdot \frac{\partial^2 F}{\partial x^2}(v) \cdot z \leq 0 \quad \text{for all } v \in A \quad \text{and for all } z \in \mathbb{R}^n \quad \text{such that } \frac{\partial F}{\partial x}(v) \cdot z = 0. \]
In this exercise, you will prove that \( i \) implies \( iii \). Here are some hints: It's probably easiest to prove the contrapositive; that is, not \( iii \) implies not \( i \). To this end, assume that there exists \( \nu \in A \) and \( z \in \mathbb{R}^n \) such that

\[
\frac{\partial F}{\partial x}(\nu) \cdot z = 0 \quad \text{but} \quad z' \cdot \frac{\partial^2 F}{\partial x^2}(\nu) \cdot z > 0.
\]

It remains to show that \( F(\cdot) \) is not quasi-concave. To do this, first note that continuity implies that there exists an \( \varepsilon > 0 \) such that

\[
z' \cdot \frac{\partial^2 F}{\partial x^2}(\nu) \cdot z > 0 \quad \text{for all} \quad \tilde{\nu} \in N_\varepsilon(\nu).
\]

Now consider the value of \( F(\cdot) \) at points a "small" distance to either side of \( \nu \) in the direction \( z \). Use a Taylor series expansion to establish a violation of \( i \).

3. (part of Simon and Blume, Theorem 21.20) Let \( F : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) be differentiable and quasi-concave. Use the second derivative characterization of quasi-concavity from the theorem in problem 2 to show that

\[
\text{det}
\begin{bmatrix}
0 & F_1 & F_2 \\
F_1 & F_{11} & F_{12} \\
F_2 & F_{12} & F_{22}
\end{bmatrix} \geq 0.
\]

4. A firm uses \( n \) inputs in quantities \( x_1, x_2, \ldots, x_n \) to produce output in quantity \( Q \). Technology is summarized by the production function \( Q = F(x_1, x_2, \ldots, x_n) \). The revenue that the firm gets by selling output \( Q \) is given by the revenue function, \( R(Q) \). Inputs are purchased at prices; denoted \( w_1, w_2, \ldots, w_n \); that are constant with respect to the quantities of the inputs. Both the production and revenue functions are differentiable.

Consider the problem of maximizing profit by choice of the \( n \) input quantities. Suppose that, for a given initial value of the input price vector, the problem has a regular solution at which the profit maximizing employment levels are locally differentiable functions of \( w \): \( x_i^*(w) \), for \( i = 1, 2, \ldots, n \). (Note: A "regular solution" is terminology sometimes used to describe a solution at which the second-order sufficient conditions are satisfied.) Show that the \( n \times n \) matrix with row \( i \), column \( j \) element \( \frac{\partial x_i^*}{\partial w_j} \) is negative definite. Note that this result is independent of the specific forms of the production or revenue functions.
(Hint: Proving this result is a relatively simple matter of differentiating the first-order necessary conditions. You'll have to use the fact that the inverse of a symmetric, negative definite matrix is negative definite. Give a proof of this fact too.)

Figures for problem 1: