1. Let $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}$ be a differentiable function of a vector of arguments $(x, z, \theta)$, where $x$ and $z$ are vectors of choice variables of dimensions $n \geq 1$ and $m \geq 1$, respectively, and $\theta$ is a scalar parameter. Consider the problem:

$$\max_{w.r.t. x, z; \text{ for given } \theta} F(x, z, \theta).$$

Assume that the problem has a strict global solution for each value of $\theta$. Denote the optimal values of $x$ and $z$ by $x^*(\theta)$ and $z^*(\theta)$ (assumed differentiable) and define the value function: $F^*(\theta) \equiv F(x^*(\theta), z^*(\theta), \theta)$. Now consider the problem:

$$\max_{w.r.t. x, z; \text{ for given } \theta} F(x, z, \theta).$$

Again assume that there is a strict global solution for all $z$ and $\theta$ denoted $x(\theta, z)$ (again assumed differentiable) and define the value function: $F(\theta, z, \theta) \equiv F(x(\theta, z), z, \theta)$. It is obvious that $x^*(\theta) = x(z^*(\theta), \theta)$ and $F^*(\theta) = F(z^*(\theta), \theta)$.

a. Use the envelope theorem to show

$$\frac{dF^*}{d\theta}(\theta) = \frac{\partial \hat{F}}{\partial \theta}(z^*(\theta), \theta).$$

b. Show that

$$\frac{d^2F^*}{d\theta^2}(\theta) \geq \frac{\partial^2 \hat{F}}{\partial \theta^2}(z^*(\theta), \theta).$$

(Hint: For any $\theta_0$ and $\theta_1$, we have $F^*(\theta_1) = \hat{F}(z^*(\theta_1), \theta_1) \geq \hat{F}(z^*(\theta_0), \theta_1)$. Viewing $F^*(\theta_1)$ and $\hat{F}(z^*(\theta_0), \theta_1)$ as functions of $\theta_1$ for fixed $\theta_0$, write down second-order Taylor series expansions for them taking $\theta_0$ as the expansion point.)

c. Use the results of part b and Hotelling's lemma (see, for example, Simon and Blume, Theorem 22.11) to prove that a competitive firm's "long-run" supply curve is at least as elastic as its "short-run" supply curve.
2. This problem is closely related to the previous one. Let \( F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R} \) and \( g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R} \) be differentiable functions of a vector of arguments \((x, z, \theta)\), where \( x \) and \( z \) are vectors of choice variables of dimensions \( n \geq 1 \) and \( m \geq 1 \), respectively, and \( \theta \) is a scalar parameter. We can think of the elements of \( x \) as choice variables that can be adjusted in the "long-run" or the "short-run," while the elements of \( z \) are choice variables that can be adjusted only in the long-run and are fixed in the short-run. First consider the long-run, equality-constrained, maximization problem:

\[
\max \quad F(x, z, \theta) \quad \text{subject to} \quad g(x, z, \theta) = b,
\]

where \( b \) is a scalar constraint constant. Assume that the problem has a strict global solution for each value of \( \theta \). Denote the optimal values of \( x \) and \( z \) by \( x^*(\theta) \) and \( z^*(\theta) \) (assumed differentiable) and define the long-run value function:

\[
F^*(\theta) \equiv F(x^*(\theta), z^*(\theta), \theta).
\]

Now consider the short-run, equality-constrained, maximization problem:

\[
\max \quad F(x, z, \theta) \quad \text{subject to} \quad g(x, z, \theta) = b.
\]

Again assume that there is a strict global solution for all \( z \) and \( \theta \) denoted \( \hat{x}(z, \theta) \) (again assumed differentiable) and define the short-run value function:

\[
\hat{F}(z, \theta) \equiv F(\hat{x}(z, \theta), z, \theta).
\]

It is obvious that \( x^*(\theta) = \hat{x}(z^*(\theta), \theta) \) and \( F^*(\theta) = \hat{F}(z^*(\theta), \theta) \).

a. Use the envelope theorem to establish the "first-order envelope property":

\[
\frac{dF^*}{d\theta}(\theta) = \frac{\partial \hat{F}}{\partial \theta}(z^*(\theta), \theta).
\]

b. Review the proof of the "second-order envelope property,"

\[
\frac{d^2F^*}{d\theta^2}(\theta) \geq \frac{\partial^2 \hat{F}}{\partial \theta^2}(z^*(\theta), \theta),
\]

given in part b of the previous problem. This proof carries over to the present case of an equality-constrained maximization problem except that we have to add the assumption that, for any \( z \) and \( \theta \), we can find an \( x \) that satisfies the constraint. Where is this additional assumption used in the proof?

(Note: The first-order envelope property says that the long-run and short-run value functions are tangent at the point they have in common. The second-order envelope property says that, at this tangency point, the long-run value function is at least as convex as the short-run value function.)
c. Use the result of part b, the expenditure function from consumer theory, and its relation to Hicksian (compensated) demand functions (discussed in Mas-Colell, Whinston, and Green, section 3.E and Proposition 3.G.1, for example) to prove that a consumer’s Hicksian demand for a good is at least as responsive to changes in own-price in the long-run (when quantities of all goods can be chosen freely) as in the short-run (when the quantities of some goods are fixed).

(Note: This result is called the LeChatelier-Samuelson principle.)

3. This problem is closely related to the previous two. Let \( F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \rightarrow \mathbb{R} \) be differentiable functions of a vector of arguments \((x, z, \theta)\), where \(x\) and \(z\) are vectors of choice variables of dimensions \(n \geq 1\) and \(m \geq 1\), respectively, and \(\theta\) is a scalar parameter. Consider the problem:

\[
\min_{w.r.t. \ x, \ z; \ for \ given \ \theta} F(x, z, \theta) \quad \text{subject to} \quad g(x, z, \theta) = 0,
\]

Assume that the problem has a strict global solution for each value of \(\theta\). Denote the optimal values of \(x\) and \(z\) by \(x^*(\theta)\) and \(z^*(\theta)\) (assumed differentiable) and define the value function: \(F^*(\theta) = F(x^*(\theta), z^*(\theta), \theta)\). Now consider the problem:

\[
\min_{w.r.t. \ x; \ for \ given \ \theta, \ z} F(x, z, \theta) \quad \text{subject to} \quad g(x, z, \theta) = 0.
\]

Again assume that there is a strict global solution for all \(z\) and \(\theta\) denoted \(\hat{x}(z, \theta)\) (again assumed differentiable) and define the value function: \(\hat{F}(z, \theta) = F(\hat{x}(z, \theta), z, \theta)\).

State and prove the second-order envelope property relevant to these two problems.

Every principles of microeconomics textbook contains a graph of a family of \(U\)-shaped short-run average cost curves and the associated long-run average cost curve. How is the second-order envelope property evident in the appearance of these graphs?

4. Consider the following expenditure minimization problem for a consumer with utility function \(u(x_1, x_2)\):

\[
\min_{w.r.t. \ x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{subject to} \quad u(x_1, x_2) = U.
\]

Assume that \(u(\cdot)\) is differentiable with strictly positive partial derivatives throughout \(\mathbb{R}^2_+\). Further assume that: given initial values for prices, \(p_1\) and \(p_2\), and utility, \(U\); we have strictly positive solutions, \(x_1^*\) and \(x_2^*\) at which the second-order sufficient conditions for the problem are satisfied. (These assumptions mean that it is safe to ignore the non-
negativity restrictions on $x_1$ and $x_2$, and to treat the utility constraint as an equality, as we have done in the statement of the problem above. Denote the solutions as functions (assumed differentiable) of the parameters: $x_1^*(p_1, p_2; U)$ and $x_2^*(p_1, p_2; U)$. Define the expenditure function: $E(p_1, p_2; U) = p_1 x_1^*(p_1, p_2; U) + p_2 x_2^*(p_1, p_2; U)$.

a. Prove that $\frac{\partial x_1^*}{\partial p_2} = \frac{\partial x_2^*}{\partial p_1}$ with an approach that uses the envelope theorem and Young's theorem. (Young's theorem says that the second-order cross partial derivatives of differentiable functions are independent of the order in which the derivatives are taken.)

b. Prove the same result stated in part a with an approach that applies the implicit function theorem to the first-order necessary conditions for the expenditure minimization problem.