1. There are two goods: 1 and 2. Both goods are available at two different kinds of stores: regular stores and bargain stores. Prices for goods 1 and 2 at regular stores are $p_1$ and $p_2$ respectively; at bargain stores, prices are $q_1$ and $q_2$. Of course, bargain store prices are lower than regular store prices. Moreover, the bargain store price advantage is greater, in absolute terms, on good 1 than on good 2: $p_1 - q_1 > p_2 - q_2 > 0$.

Let $x_i$, for $i = 1$ and 2, denote the quantity of good $i$ purchased at regular stores. Similarly, let $y_i$, for $i = 1$ and 2, denote bargain store purchases of good $i$. Utility depends just on the total consumption of good 1 and the total consumption of good 2: $U(x_1 + y_1, x_2 + y_2)$. The marginal utilities of both goods are strictly positive throughout the commodity space. Regular store and bargain store purchases of each good must be non-negative, and total expenditure on goods 1 and 2 cannot exceed positive income $I$:

$$x_1 \geq 0, \ x_2 \geq 0, \ y_1 \geq 0, \ y_2 \geq 0, \ p_1 x_1 + p_2 x_2 + q_1 y_1 + q_2 y_2 \leq I.$$ 

One final constraint: Total combined quantity of goods 1 and 2 purchased from bargain stores cannot exceed a positive limit, $K$: $y_1 + y_2 \leq K$.

(If you want to think in terms of a concrete example, imagine good 1 being whiskey and good 2 being gin. Each can be purchased in regular stores and, at lower prices, in duty-free stores in international airports. But my country’s custom regulations limit the total number of bottles of whiskey and gin that I can bring home, duty-free.)

(a) Write the Lagrangian for this inequality and non-negativity constrained utility maximization problem.

(b) Write the Kuhn-Tucker conditions.

(c) Use the Kuhn-Tucker conditions to prove the following (where starred values of choice variables denote optimal values):

i. $x_1^* + x_2^* > 0$ implies $y_1^* + y_2^* = K$.

ii. $y_2^* > 0$ implies $x_1^* = 0$. 
d. Suppose that \( x_1^* = 0, x_2^* > 0, y_1^* = K, \text{ and } y_2^* = 0 \). Use the Kuhn-Tucker conditions to prove the following relationship:

\[
\frac{q_1}{p_2} \leq \frac{U_1(x_1^* + y_1^*, x_2^* + y_2^*)}{U_2(x_1^* + y_1^*, x_2^* + y_2^*)} \leq \frac{p_1}{p_2}.
\]

Provide an economic interpretation of this result.

(Important note: The function \( U(\cdot) \) has two arguments and, therefore, two first-order partial derivatives. I suggest you follow my example and use the notations "\( U_1(\cdot) \)" and "\( U_2(\cdot) \)" for these. Avoid use of ambiguous notations like "\( U_{x_1}(\cdot) \)" or "\( U_{y_1}(\cdot) \).")

2. A sum of \( C \) dollars is available for allocation among \( n \) investment projects. If the non-negative amount \( x_i \) is allocated to project \( i \), for \( i = 1, 2, \ldots, n \), the expected return from the portfolio will be

\[
\sum_{i=1}^{n} \left[ \alpha_i x_i - \frac{1}{2} \beta_i x_i^2 \right],
\]

where the \( \alpha_i \)'s and \( \beta_i \)'s are positive constants. The \( x_i \)'s are to be chosen to maximize the above sum subject to the constraints that each \( x_i \) be non-negative and that the sum of the \( x_i \)'s be no more than \( C \). Define the following:

\[
H = \sum_{i=1}^{n} \alpha_i / \beta_i \quad \text{and} \quad K = \sum_{i=1}^{n} 1 / \beta_i.
\]

Write down the Kuhn-Tucker conditions for this problem and use them to prove each of the following propositions (where starred values of the \( x_i \)'s denote optimal values):

a. \( \sum_{i=1}^{n} x_i^* < C \quad \text{if and only if} \quad C > H. \)

b. If \( x_j^* > 0 \) and \( x_k^* = 0 \) then \( \alpha_k < \alpha_j \).

c. If \( x_j^* > 0 \) then \( \alpha_j > (H - C)/K. \)

d. If \( H < C \) then \( x_i^* > 0 \) for all \( i = 1, 2, \ldots, n \).
e. If $H \geq C$ and $\alpha_i > (H - C)/K$ for all $i = 1, 2, \ldots, n$; then $x_i^* > 0$ for all $i = 1, 2, \ldots, n$. 

(Hint: Several of these implications might be more easily proved by proving the contrapositive rather than by proving the implication directly. Special hint on part e: Assume that there is a $j$ such that $x_j^* = 0$. By part b, we can take $j$ so that $\alpha_j \leq \alpha_i$ for all $i \neq j$. Show that $x_i^* \leq (\alpha_i - \alpha_j)/\beta_i$ for all $i = 1, 2, \ldots, n$. Then sum over $i$ to obtain a contradiction.)