Exam #1 Solution Outline

1. a. True.
   
b. False. Change "if and only if" to "if" or change "strictly concave" to "concave" and change "negative definite" to "negative semi-definite."
   
c. False. Change "positive definite" to "positive semi-definite" or change "implies that" to "is implied by" and change 
   
   \[
   \frac{\partial^2 F}{\partial x^2}(x^*) \text{ is positive definite} \]
   
   to
   
   \[
   \frac{\partial^2 F}{\partial x^2}(x^*) \text{ is positive definite and } \frac{\partial F}{\partial x}(x^*) = 0
   \]
   
d. True. e. True.

2. **Proof:** (by contradiction) Assume that \( x^* \) is a strict global solution to the problem. That is, assume \( x^* \in U \) and
   
   \[ F(x^*) < F(x) \text{ for all } x \in U, x \neq x^* \] (*)
   
   \( U \) open implies there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(x^*) \subseteq U \). (Intuitively, this means we can find two points "on opposite sides" of \( x^* \) such that both points are in \( U \). For example . . )
   
   Take \( i_1 = (1, 0, 0, \ldots, 0) \) and define \( u = x^* + \frac{\varepsilon}{2} i_1 \) and \( v = x^* - \frac{\varepsilon}{2} i_1 \). Then \( u, v \in U \) and by quasi-concavity of \( F(\cdot) \):
   
   \[ F(x^*) = F\left(\frac{1}{2} u + \frac{1}{2} v\right) \geq \min\{F(u), F(v)\}, \text{ but this contradicts } (*)\]
   
   Note: A stronger claim is also valid and could be proved with very slight changes to the proof above. Under the stated hypotheses:
   
   \[ \min_{w.r.t. x} F(x) \text{ subject to } x \in U \text{ cannot have a strict local solution.} \]
3. a. \( L(x_1, x_2; \lambda) = x_1^2 + x_2^2 + \lambda \left(1 - x_1^2 - \frac{1}{4} x_2^2\right) \)

\[
\frac{\partial L}{\partial x_1} = 2x_1 - 2\lambda x_1 = 0, \quad \frac{\partial L}{\partial x_2} = 2x_2 - \frac{1}{2} \lambda x_2 = 0, \quad \frac{\partial L}{\partial \lambda} = 1 - x_1^2 - \frac{1}{4} x_2^2 = 0
\]

b. Solutions: \((x_1^*, x_2^*; \lambda^*) = (0, 2, 4), (0, -2, 4), (1, 0, 1), \) and \((-1, 0, 1)\)

c. \( \mathcal{H} = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{12} & L_{22} \end{bmatrix} (\lambda', x') = \begin{bmatrix} 0 & 2x_1 & \frac{1}{2} x_2 \\ 2x_1 & 2 - 2\lambda & 0 \\ \frac{1}{2} x_2 & 0 & 2 - \frac{1}{2} \lambda \end{bmatrix} (\lambda', x') \)

\[\det \mathcal{H} = -2x_1^2(4 - \lambda) - \frac{1}{2} x_2^2(1 - \lambda)\]

(Note: SOSC for max: \( \det \mathcal{H} > 0 \). SOSC for min: \( \det \mathcal{H} < 0 \).)

\[\det \mathcal{H}_{(x_1, x_2; \lambda)}(0, \pm 2, 4) = 6 \quad \text{and} \quad \det \mathcal{H}_{(x_1, x_2; \lambda)}(\pm 1, 0, 1) = -6\]

So \((x_1^*, x_2^*) = (0, 2)\) and \((0, -2)\) are strict local maxima,

and \((x_1^*, x_2^*) = (1, 0)\) and \((-1, 0)\) are strict local minima.

Note: An equivalent problem involves maximizing or minimizing, subject to the same constraint, the following strictly increasing transformation of the original objective function:

\[\bar{F}(x_1, x_2) = (x_1^2 + x_2^2)^{\frac{1}{2}}.\]

\(\bar{F}(x_1, x_2)\) is the Euclidean distance from \((x_1, x_2)\) to the origin. So the problem involves finding the points on the ellipse:

\[x_1^2 + \frac{1}{4} x_2^2 = 1\]

that are closest to, and farthest away from, the origin.