Exam #1 Solution Outline


d. False.   "... the gradient vector is perpendicular to the level curve."

e. False.

\[
\frac{\partial^2 F}{\partial x^2}(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}; \text{ negative definite, hence SOSC is satisfied.}
\]

2. \( F: \mathbb{R}^n \to \mathbb{R} \) is strictly quasi-concave and \( x^* \in \mathbb{R}^n \) is a local solution to \( \max_{\text{w.r.t. } x} F(x) \).

Then \( x^* \) is a strict global solution.

**Proof:** (by contradiction) Suppose that \( x^* \) is not a strict global solution. That is, suppose that there exists \( \hat{x} \in \mathbb{R}^n, \hat{x} \neq x^* \), such that \( F(\hat{x}) \geq F(x^*) \).

\( F(\cdot) \) strictly quasi-concave implies that, for all \( h \in (0,1) \),

\[
F(hx^* + (1-h)\hat{x}) > \min\{F(x^*), F(\hat{x})\} = F(x^*).
\]

\( (***) \)

For \( h \) sufficiently close to 1, however, \( hx^* + (1-h)\hat{x} \in B_\epsilon(x^*) \).

So \( (*) \) and \( (**) \) are a contradiction.

3. \[ \max_{\text{w.r.t. } x_1, x_2, \ldots, x_n} \pi(x_1, x_2, \ldots, x_n; p; w_1, w_2, \ldots, w_n), \]

where \( \pi(\cdot) \equiv pF(x_1, x_2, \ldots, x_n) - \sum_{i=1}^{n} w_i x_i \).

Solutions (assumed differentiable): \( x^*_i(p; w_1, w_2, \ldots, w_n) \quad i = 1, 2, \ldots, n. \)

Define the value function: \( \pi^*(p; w_1, w_2, \ldots, w_n) \equiv \pi(x^*_1(\cdot), x^*_2(\cdot), \ldots, x^*_n(\cdot); p; w_1, w_2, \ldots, w_n) \)
By the envelope theorem:
\[
\frac{\partial \pi^*}{\partial w_i} = \frac{\partial \pi}{\partial w_i} (x^*; p; w) = -x_i^* (p; w) \quad \text{for } i = 1, 2, \ldots, n.
\]

For \(i \neq j = 1, 2, \ldots, n\):
\[
\frac{\partial x_i^*}{\partial w_j} = -\frac{\partial^2 \pi^*}{\partial w_j \partial w_i} = -\frac{\partial^2 \pi^*}{\partial w_i \partial w_j} \quad \text{(by Young's Theorem)}
\]
\[
= \frac{\partial x_i^*}{\partial w_i}.
\]