Exam #2 Solution Outline

1. a. True.  
   b. True.  
   c. True  
   d. False. Change "\( \frac{\partial F}{\partial x_i}(x^*) \geq 0 \)" to "\( \frac{\partial F}{\partial x_i}(x^*) \leq 0 \)."
   e. True.

2. a. \[
\max_{x_1, x_2} U(x_1, x_2) \quad \text{subject to} \quad p_1 x_1 + p_2 x_2 \leq M, \quad r_1 x_1 + r_2 x_2 \leq R, \quad x_1, x_2 \geq 0
\]
   b. \[
M(x_1, x_2; \lambda, \mu) = U(x_1, x_2) + \lambda(M - p_1 x_1 - p_2 x_2) + \mu(R - r_1 x_1 - r_2 x_2)
\]
   
   (I) \[
\frac{\partial M}{\partial x_1}(x^*; \lambda^*, \mu^*) = U_1(x^*) - \lambda^* p_1 - \mu^* r_1 \leq 0, \quad x_1^* \geq 0, \quad \frac{\partial M}{\partial x_1}(x^*; \lambda^*, \mu^*) \cdot x_1^* = 0
\]
   (II) \[
\frac{\partial M}{\partial x_2}(x^*; \lambda^*, \mu^*) = U_2(x^*) - \lambda^* p_2 - \mu^* r_2 \leq 0, \quad x_2^* \geq 0, \quad \frac{\partial M}{\partial x_2}(x^*; \lambda^*, \mu^*) \cdot x_2^* = 0
\]
   (III) \[
\frac{\partial M}{\partial \lambda}(x^*; \lambda^*, \mu^*) = M - p_1 x_1^* - p_2 x_2^* \geq 0, \quad \lambda^* \geq 0, \quad \frac{\partial M}{\partial \lambda}(x^*; \lambda^*, \mu^*) \cdot \lambda^* = 0
\]
   (IV) \[
\frac{\partial M}{\partial \mu}(x^*; \lambda^*, \mu^*) = R - r_1 x_1^* - r_2 x_2^* \geq 0, \quad \mu^* \geq 0, \quad \frac{\partial M}{\partial \mu}(x^*; \lambda^*, \mu^*) \cdot \mu^* = 0
\]
   c. i. **Proof:** \( x_1^*, x_2^* > 0 \) and complementary slackness in (I) and (II) imply that the marginal conditions in (I) and (II) hold as equalities. \( p_1 x_1^* + p_2 x_2^* < M \) and complementary slackness in (III) imply \( \lambda^* = 0 \). \( U_1(x^*) > 0 \) and marginal condition in (I) imply \( \mu^* > 0 \) . . . plus marginal condition in (IV) implies \( r_1 x_1^* + r_2 x_2^* = R \).
   \[
U_1(x^*) = \mu^* r_1, \quad U_2(x^*) = \mu^* r_2. \quad \text{Taking ratio:} \quad \frac{MRS_{2/1}(x^*)}{U_2(x^*)} = \frac{\mu^*}{r_1} = \frac{r_2}{r_2}. \quad \text{Q.E.D.}
\]
   c. ii. **Proof:** \( x_2^* = 0 \) and marginal condition in (III) implies \( p_1 x_1^* \leq M \). So
\[ x^*_i \leq \frac{M}{p_i} < \frac{R}{r_i} \] which implies \( r_ix^*_i < R \).

But then complementary slackness in (IV) implies \( \mu^* = 0 \). \( x^*_i > 0 \) and (I) implies \( U_1(x^*) = \lambda^* p_1 \).

\( U_1(x^*) > 0 \) implies \( \lambda^* > 0 \). With complementary slackness in (III), this implies \( p_ix^*_i = M \), so \( x^*_i = M/p_1 \). \( x^*_2 = 0 \) and (II) implies \( U_2(x^*) \leq \lambda^* p_2 \).

Taking ratio: \( MRS_{2/1}(x^*) = \frac{U_1(x^*)}{U_2(x^*)} \leq \frac{p_1}{p_2} \)

3. **Proof:** Pick \( n_1 \) such that \( |x^n_1 - x^*| < \frac{1}{1} \). (This is possible because \( x^* \) is an accumulations point.) Set \( y_1 = x^n_1 \). Pick \( n_2 \) such that \( n_2 > n_1 \) and \( |x^n_2 - x^*| < \frac{1}{2} \). (This is possible because, since \( x^* \) is an accumulation point, there are infinitely many \( n \) such that \( |x^n - x^*| < \frac{1}{2} \). By requiring that the chosen \( n, n_2 \), be greater than \( n_1 \) we exclude, at most, only finitely many of them.) Set \( y_2 = x^n_2 \).
Continuing in this way for \( k = 3, 4, 5, \ldots \) pick \( n_k \) such that \( n_k > n_{k-1} \) and 
\[ |x_{n_k} - x^*| < \frac{1}{k} \]. Set \( y_k = x_{n_k} \). Then \( \{y_k\}^\infty_{k=1} \) is a subsequence of \( \{x_n\}^\infty_{n=1} \) and 
\( \{y_k\}^\infty_{k=1} \to x^* \). (For all \( \varepsilon > 0 \), there exists \( K \) such that \( \frac{1}{K} < \varepsilon \). But then 
\[ |y_k - x^*| < \frac{1}{K} < \varepsilon \] for all \( k \geq K \).) Q.E.D.

4. Differentiating (***):

\[
\frac{\partial z^*_j}{\partial a_j}(a_1, a_2, f) = \frac{\partial x^*_i}{\partial a_j}(a_1, a_2, G^*(a_1, a_2, f)) + \frac{\partial x^*_i}{\partial b}(a_1, a_2, G^*(a_1, a_2, f)) \cdot \frac{\partial G^*}{\partial a_j}(a_1, a_2, f) \tag{1}
\]

Now consider the problem: \( \min_{w.r.t. z_1, z_2} a_1z_1 + a_2z_2 \) subject to \( F(z_1, z_2) = f \)

with solutions: \( z^*_1(a_1, a_2, f), \ z^*_2(a_1, a_2, f) \)

value function: \( G^*(a_1, a_2, f) = a_1z^*_1 + a_2z^*_2 \)

and Lagrangian: \( L(z_1, z_2; a_1, a_2, f) = a_1z_1 + a_2z_2 + \lambda(f - F(z_1, z_2)) \)

By the envelope theorem:

\[
\frac{\partial G^*}{\partial a_j}(a_1, a_2, f) = \frac{\partial L}{\partial a_j}(z^*_1, z^*_2; a_1, a_2, f) = z^*_j(a_1, a_2, f)
\]

Substitute into (1) and rearrange:

\[
\frac{\partial x^*_i}{\partial a_j}(a_1, a_2, G^*(a_1, a_2, f)) = \frac{\partial z^*_j}{\partial a_j}(a_1, a_2, f) - \frac{\partial x^*_i}{\partial b}(a_1, a_2, G^*(a_1, a_2, f)) \cdot z^*_j(a_1, a_2, f)
\]

Evaluate with \( f \equiv F^*(a_1, a_2, b) \) and use duality results:

\[ z^*_j(a_1, a_2, f) = x^*_j(a_1, a_2, G^*(a_1, a_2, f)) \] and \( G^*(a_1, a_2, F^*(a_1, a_2, b)) = b \).

The result is:

\[
\frac{\partial x^*_i}{\partial a_j}(a_1, a_2, b) = \frac{\partial z^*_j}{\partial a_j}(a_1, a_2, F^*(a_1, a_2, b)) - \frac{\partial x^*_i}{\partial b}(a_1, a_2, b) \cdot x^*_j(a_1, a_2, b) \quad \text{for } i, j = 1, 2.
\]

Q.E.D.