Econ 600
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Exam #2 Solution Outline

1. a. True.   b. True.   c. True.

d. False. For each $i = 1, 2, \ldots, n$: $\frac{\partial F}{\partial x_i}(x^*) - \lambda^* \frac{\partial g}{\partial x_i}(x^*) \le 0.$

e. True.

2. a. $L(x_1, x_2; \lambda) = x_2 e^{x_1} + \lambda(M - p_1 x_1 - p_2 x_2)$

I. $\frac{\partial L}{\partial x_1} = x_2 e^{x_1} - p_1 \lambda \le 0, \quad x_1 \ge 0, \quad (x_2 e^{x_1} - p_1 \lambda) \cdot x_1 = 0$

II. $\frac{\partial L}{\partial x_2} = e^{x_1} - p_2 \lambda \le 0, \quad x_2 \ge 0, \quad (e^{x_1} - p_2 \lambda) \cdot x_2 = 0$

III. $\frac{\partial L}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 \ge 0, \quad \lambda \ge 0, \quad (M - p_1 x_1 - p_2 x_2) \cdot \lambda = 0$

b. Marginal condition in II. requires $\lambda^* > 0$. Combined with complementary slackness condition in III., this gives:

$p_1 x_1^* + p_2 x_2^* = M$ (budget constraint binding)

With budget constraint binding, cannot have $x_1^* = x_2^* = 0$.

Possible to have $x_1^* > 0, x_2^* = 0$?

Then complementary slackness condition in I. implies $\lambda^* = 0$, but this violates marginal condition in II.

Possible to have $x_1^* = 0, x_2^* > 0$?

From I.: $x_2^* - p_1 \lambda^* \le 0$

From II.: $1 - p_2\lambda^* = 0 \quad \Rightarrow \quad \lambda^* = \dfrac{1}{p_2}$

So $x_2^* \leq \dfrac{p_1}{p_2}$. Budget constraint $\Rightarrow x_2^* = \dfrac{M}{p_2}$, requiring $M \leq p_1$.

Possible to have $x_1^* > 0, x_2^* > 0$?

I. $\Rightarrow \quad x_2^* e^{x_1^*} - p_1\lambda^* = 0$

II. $\Rightarrow \quad e^{x_1^*} - p_2\lambda^* = 0$

Taking ratio: $x_2^* = \dfrac{p_1}{p_2}$. From budget constraint: $x_1^* = \dfrac{M - p_1}{p_1} > 0$, requiring $p_1 < M$. (Note also: $\lambda^* > 0$.)

Summary: Two types of solutions are possible:

When $M \leq p_1$: $x_1^* = 0, x_2^* = \dfrac{M}{p_2} > 0, \; p_1x_1^* + p_2x_2^* = M$.

When $M > p_1$: $x_1^* = \dfrac{M - p_1}{p_1} > 0, x_2^* = \dfrac{p_1}{p_2} > 0, \; p_1x_1^* + p_2x_2^* = M$.

3. <u>Proof:</u> Form a sequence of distinct points in *B*: $\{x_n\}_{n=1}^{\infty}$. (For each *n*, can choose $x_n \neq x_m \; \textit{for all} \; m = 1, 2, \ldots, n-1$, because *B* is infinite.)

$\{x_n\}_{n=1}^{\infty}$ is a sequence in a compact subset of $\Re^n$. By Bolzano-Weierstrass, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence:

$$\{x_{n_k}\}_{k=1}^{\infty} \to x.$$

Claim: *x* is a cluster point of *B*.

x is the limit of a sequence in a compact (hence closed) set *B* so $x \in B$. For each $\varepsilon > 0$, there exists *K* such that

$$x_{n_k} \in B_\varepsilon(x) \quad \textit{for all} \; k \geq K.$$

Because the x_{n_k}'s are all distinct, at least one is different from *x*. Q.E.D.

4. (Graph follows)

a. Features of graph:

(*i*.) All three functions have same value at $p = p^0$;

$$\pi^*(p^0) = \pi(p^0; y^*(p^0), x_1^*(p^0), x_2^*(p^0)) \quad \text{(by definition of LR value function)}$$

$$\pi^s(p^0; x_2^*(p^0)) = \pi(p^0; y^s(p^0; x_2^*(p^0)), x_1^s(p^0; x_2^*(p^0)), x_2^*(p^0))$$

(by definition of SR value function)

$$= \pi(p^0; y^*(p^0), x_1^*(p^0), x_2^*(p^0))$$

(because $y^s(p^0; x_2^*(p^0)) = y^*(p^0)$ and $x_1^s(p^0; x_2^*(p^0)) = x_1^*(p_0)$. Same result when optimizing in 2 steps or 1.)

(*ii*.) $\pi(p; y^*(p^0), x_1^*(p^0), x_2^*(p^0)) = p \cdot y^*(p^0) - x_1^*(p^0) - x_2^*(p^0)$

Linear in *p* with slope $y^*(p^0)$.

(*iii*.) Curvatures reflect: For $p \neq p^0$:

$$\pi(p; y^*(p^0), x_1^*(p^0), x_2^*(p^0)) \leq \pi^s(p; x_2^*(p^0)) \leq \pi^*(p)$$

because removal of a constraint cannot reduce profit.

b. The curvatures of the three functions imply:

$$\frac{d^2}{dp^2}\pi^*(p^0) \geq \frac{\partial^2}{\partial p^2}\pi^s(p^0; x_2^*(p^0)) \geq \frac{\partial^2}{\partial p^2}\pi(p^0; y^*(p^0), x_1^*(p^0), x_2^*(p^0)) = 0 \quad (*)$$

The Lagrangians for the SR and LR problems are the same -- the only difference is whether x_2 is a choice variable or a parameter:

$$L(y, x_1, x_2; p; \lambda) = p \cdot y - x_1 - x_2 + \lambda(y - f(x_1, x_2))$$

By envelope theorem:

$$\frac{d}{dp}\pi^*(p) = \frac{\partial}{\partial p}L(\text{"*" } values) = y^*(p)$$

$$\frac{\partial}{\partial p}\pi^s\left(p;x_2^*\left(p^0\right)\right)=\frac{\partial}{\partial p}L\left("s"\ values\right)=y^s\left(p;x_2^*\left(p^0\right)\right)$$

Taking a second derivative w.r.t. *p* and evaluating at $p = p^0$:

$$\frac{d^2}{dp^2}\pi^*\left(p^0\right)=\frac{d}{dp}y^*\left(p^0\right)$$

$$\frac{\partial^2}{\partial p^2}\pi^s\left(p^0;x_2^*\left(p^0\right)\right)=\frac{\partial}{\partial p}y^s\left(p^0;x_2^*\left(p^0\right)\right)$$

From inequality (*):

$$\frac{d}{dp}y^*\left(p^0\right)\geq\frac{\partial}{\partial p}y^s\left(p^0;x_2^*\left(p^0\right)\right)\geq 0$$ (Supply more elastic in LR.)