Econ 600
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Homework Solution Outline

1. a. <u>Proof:</u> $\inf B \in L(B)$ so

$\inf B \le b_i$ for all $i \in I$.

$\le a_i$ for all $i \in I$. This implies that $\inf B \in L(A)$.

$\inf A \ge l$ for all $l \in L(A)$, therefore $\inf A \ge \inf B$. Q.E.D.

b. <u>Proof:</u> First show that: $\inf\{\alpha c_i : i \in I\} = \alpha \inf C$.

$\inf C \in L(C)$ so $\inf C \le c_i$ for all $i \in I$.

$\alpha \inf C \le \alpha c_i$ for all $i \in I$. So $\alpha \inf C \in L(\{\alpha c_i : i \in I\})$.

Now suppose that $\alpha \inf C$ is NOT the greatest lower bound for $\{\alpha c_i : i \in I\}$.
Then there exists l^* such that

$\alpha \inf C < l^* \le \alpha c_i$ for all $i \in I$. Dividing by $\alpha > 0$:

$\inf C < l^* / \alpha \le c_i$ for all $i \in I$.

(This is a contradiction because $\inf C$ is the greatest lower bound for C.)

Therefore $\inf\{\alpha c_i : i \in I\} = \alpha \inf C$. Q.E.D.
(And, obviously: $\inf\{\beta d_i : i \in I\} = \beta \inf D$.)

Now show that $\inf\{e_i + f_i : i \in I\} \ge \inf E + \inf F$,
where $E = \{e_i : i \in I\}$ and $F = \{f_i : i \in I\}$ are bounded below.

$\inf E \le e_i$ for all $i \in I$

$\inf F \le f_i$ for all $i \in I$ Adding:

$\inf E + \inf F \le e_i + f_i$ for all $i \in I$.

So $\inf E + \inf F \in L(\{e_i + f_i : i \in I\})$.
Therefore $\inf\{e_i + f_i : i \in I\} \ge \inf E + \inf F$. Q.E.D.

Finally: $\inf\{\alpha c_i + \beta d_i : i \in I\} \ge \inf\{\alpha c_i : i \in I\} + \inf\{\beta d_i : i \in I\}$ (by second result above)

$= \alpha \inf C + \beta \inf D$ (by first result above) Q.E.D.

c. <u>Proof:</u> Given $u, v \in \Re^n$, $h \in [0,1]$, want to show:

$$f(hu + (1-h)v) \geq hf(u) + (1-h)f(v).$$

For h = 0 or 1, result is obvious, so take $h \in (0,1)$:

$$f(hu + (1-h)v) = \inf\{f_i(hu + (1-h)v) : i \in I\} \geq \inf\{hf_i(u) + (1-h)f_i(v) : i \in I\}$$

(because $f_i(hu + (1-h)v) \geq hf_i(u) + (1-h)f_i(v)$ by concavity of $f_i(\cdot)$, and result of part *a*)

$$\geq h\inf\{f_i(u) : i \in I\} + (1-h)\inf\{f_i(v) : i \in I\}$$

(by result of part *b*)

$$= hf(u) + (1-h)f(v)$$ Q.E.D.

2. <u>Proof:</u> Given $u, v \in \Re^m$, $h \in [0,1]$, want to show:

$$g(hu + (1-h)v) \geq hg(u) + (1-h)g(v)$$

$$g(hu + (1-h)v) = f(A(hu + (1-h)v) + b) = f(h(Au + b) + (1-h)(Av + b))$$

$$\geq hf(Au + b) + (1-h)f(Av + b)$$

(by concavity of $f(\cdot)$)

$$= hg(u) + (1-h)g(v)$$ Q.E.D.

3. The function $f : \Re^2 \to \Re$ defined by $f(x_1, x_2) = x_1 x_2$ is NOT quasi-concave.

Quasi-concavity means convexity of upper contour sets.

$(2, 2)$ *and* $(-2, -2) \in \{x \in \Re^2 : f(x) \geq 1\}$ but the convex combination:

$$\tfrac{1}{2}(2, 2) + \tfrac{1}{2}(-2, -2) = (0, 0) \notin \{x \in \Re^2 : f(x) \geq 1\}.$$

Therefore $\{x \in \Re^2 : f(x) \geq 1\}$ is not convex and $f(\cdot)$ is not quasi-concave.

4. U convex subset of $\Re^n$. $f: U \to \Re$ concave and strictly quasi-concave.

Is $f(\cdot)$ necessarily strictly concave?

NO. Counterexample: $n = 1$, $U = \Re^1$ and $f: \Re \to \Re$ defined by $f(x) = x$ (or any other linear function).

Concave?: $u, v \in \Re,\ h \in [0, 1],\ f(hu + (1-h)v) = hu + (1 - hv) = hf(u) + (1-h)f(v)$.
So $f(\cdot)$ is concave.

Strictly quasi-concave?: $u, v \in \Re, u \neq v,\ h \in (0, 1),\ f(hu + (1-h)v) = hu + (1-h)v$
$> \min\{u, v\}$
$= \min\{f(u), f(v)\}$.
So $f(\cdot)$ is strictly quasi-concave.

Strictly concave?: Let $u = 0,\ v = 1,\ h = \frac{1}{2}$:
$$f(hu + (1-h)v) = \tfrac{1}{2} \ not > \tfrac{1}{2} = f(u) + (1-h)f(v)$$
So $f(\cdot)$ is NOT strictly concave.

5. U convex subset of $\Re^n$.
$$F: U \to \Re \ is \ quasi-convex \Leftrightarrow -F: U \to \Re \ is \ quasi-concave$$

<u>Proof:</u> $(\Rightarrow)$ Given $F: U \to \Re\ quasi-convex$. want to show:
$-F: U \to \Re\ quasi-concave$.

That is, for all $u, v \in U,\ h \in [0, 1]$, want to show:
$$-F(hu + (1-h)v) \geq \min\{-F(u), -F(v)\}$$

$$F(\cdot)\ quasi-convex \Rightarrow F(hu + (1-h)v) \leq \max\{F(u), F(v)\}$$
$$-F(hu + (1-h)v) \geq -\max\{F(u), F(v)\}$$
$$= \min\{-F(u), -F(v)\} \quad \text{Q.E.D.}$$

(. . . because $\max\{a, b\} = -\min\{-a, -b\}$. W.l.o.g., assume $a = \max\{a, b\}$. Then $a \geq b$, so $-a \leq -b \Rightarrow -a = \min\{-a, -b\}$. Therefore: $\max\{a, b\} = -\min\{-a, -b\}$.)

<u>Proof:</u> $(\Leftarrow)$ Similar.