Homework Solution Outline

1. \[
\max_{w.r.t. x_1, x_2, p, w_1, w_2} D(x_1, x_2; p, w_1, w_2)
\]

where \( D() \equiv \pi(x_1, x_2; p, w_1, w_2) - \pi^*(p, w_1, w_2) \).

a. FONC for local max at \( (x_1^*(p, w_1, w_2), x_2^*(p, w_1, w_2), p, w_1, w_2) \):

\[
\frac{\partial D}{\partial x_1}(*) = \frac{\partial \pi^*}{\partial x_1}(*) = 0
\]

\[
\frac{\partial D}{\partial x_2}(*) = \frac{\partial \pi^*}{\partial x_2}(*) = 0
\]

\[
\frac{\partial D}{\partial p}(*) = \frac{\partial \pi}{\partial p}(*) - \frac{\partial \pi^*}{\partial p}(p, w_1, w_2) = 0
\]

\[
\frac{\partial D}{\partial w_1}(*) = \frac{\partial \pi}{\partial w_1}(*) - \frac{\partial \pi^*}{\partial w_1}(p, w_1, w_2) = 0
\]

\[
\frac{\partial D}{\partial w_2}(*) = \frac{\partial \pi}{\partial w_2}(*) - \frac{\partial \pi^*}{\partial w_2}(p, w_1, w_2) = 0
\]

where "(*)" denotes evaluation at \( (x_1^*(p, w_1, w_2), x_2^*(p, w_1, w_2), p, w_1, w_2) \).

First two are FONC for \( \pi - \max \). Last three are envelope theorem.

b. SONC is that Hessian of \( D \) is negative semi-definite at \( (x_1^*, x_2^*; p, w_1, w_2) \). Look at last three rows and columns. (According to hint, must be negative semi-definite too.)
Since \( \pi(x_1, x_2; p, w_1, w_2) \equiv p \cdot f(x_1, x_2) - w_1 x_1 - w_2 x_2 \), we have:

\[
\frac{\partial^2 \pi}{\partial p^2} = \frac{\partial^2 \pi}{\partial p \partial w_1} = \frac{\partial^2 \pi}{\partial p \partial w_2} = \ldots = \frac{\partial^2 \pi}{\partial w_2^2} = 0.
\]

So the matrix at the top of the page becomes:

\[
\begin{bmatrix}
\frac{\partial}{\partial p} \left( \frac{\partial \pi}{\partial p} \right) & \frac{\partial}{\partial w_1} \left( \frac{\partial \pi}{\partial p} \right) & \frac{\partial}{\partial w_2} \left( \frac{\partial \pi}{\partial p} \right) \\
\frac{\partial}{\partial p} \left( \frac{\partial \pi}{\partial w_1} \right) & \frac{\partial}{\partial w_1} \left( \frac{\partial \pi}{\partial w_1} \right) & \frac{\partial}{\partial w_2} \left( \frac{\partial \pi}{\partial w_1} \right) \\
\frac{\partial}{\partial p} \left( \frac{\partial \pi}{\partial w_2} \right) & \frac{\partial}{\partial w_1} \left( \frac{\partial \pi}{\partial w_2} \right) & \frac{\partial}{\partial w_2} \left( \frac{\partial \pi}{\partial w_2} \right)
\end{bmatrix}
\]

From FONC:

\[
\frac{\partial \pi^*}{\partial p} = \frac{\partial \pi}{\partial p}(*) = f(x_1^*, x_2^*) = y^* , \quad \frac{\partial \pi^*}{\partial w_1} = \frac{\partial \pi}{\partial w_1}(*) = -x_1^*, \quad \text{and}
\]

\[
\frac{\partial \pi^*}{\partial w_2} = \frac{\partial \pi}{\partial w_2}(*) = -x_2^*\quad \Rightarrow \text{so the matrix above is:}
\]

\[
\begin{bmatrix}
-\frac{\partial y}{\partial p} & -\frac{\partial y}{\partial w_1} & -\frac{\partial y}{\partial w_2} \\
\frac{\partial x_1}{\partial p} & \frac{\partial x_1}{\partial w_1} & \frac{\partial x_1}{\partial w_2} \\
\frac{\partial x_2}{\partial p} & \frac{\partial x_2}{\partial w_1} & \frac{\partial x_2}{\partial w_2}
\end{bmatrix}
\]

Necessary and sufficient conditions for this matrix to be negative semi-definite include

the requirements that first-order principal minors be less than or equal to zero, and that

second-order principal minors be greater than or equal to zero. These are the restrictions

on comparative static derivatives stated in the problem.
2. $\pi(w_i; x_1, x_2) \equiv f(x_1, x_2) - w_i x_1 - x_2$

By the envelope theorem:

LR: $\frac{\partial \pi^*}{\partial w_1}(w_i) = \frac{\partial \pi}{\partial w_1}(w_i; x_1^*(w_i), x_2^*(w_i)) = -x_1^*(w_i)$

SR: $\frac{\partial \pi^*}{\partial w_1}(w_i, x_2) = \frac{\partial \pi}{\partial w_1}(w_i; x_1^*(w_1, x_2), x_2) = -x_1^*(w_i, x_2)$

The envelope relationship in the graph implies:

$$\frac{\partial^2 \pi^*}{\partial w_1^2}(w_i^0) \geq \frac{\partial^2 \pi^*}{\partial w_1^2}(w_i^0, x_2^*(w_i^0)) \geq \frac{\partial^2 \pi}{\partial w_1^2}(w_i, x_1^*(w_i^0), x_2^*(w_i^0)) = 0$$

This implies: $-\frac{\partial x_1^*}{\partial w_1}(w_i^0) \geq -\frac{\partial x_1^*}{\partial w_1}(w_i^0, x_2^*(w_i^0)) \geq 0$ or $\frac{\partial x_1^*}{\partial w_1}(w_i^0) \leq \frac{\partial x_1^*}{\partial w_1}(w_i^0, x_2^*(w_i^0)) \leq 0$