Homework solution outline

1. a. Lagrangians for the two problems:

\[ L_i(x; \alpha; \lambda) = F(x; \alpha) - \lambda g(x; \alpha) \]
\[ L_{ii}(x; \alpha; \lambda, \mu) = F(x; \alpha) - \lambda g(x; \alpha) - \mu h(x; \alpha) \]

FONCs for solution to (i) with \( \alpha = \alpha_0 \) include:

\[ \frac{\partial L_i}{\partial x} (x^*(\alpha_0), \alpha_0; \lambda^*(\alpha_0)) = \frac{\partial F}{\partial x} (x^*(\alpha_0), \alpha_0) - \lambda^*(\alpha_0) \frac{\partial g}{\partial x} (x^*(\alpha_0), \alpha_0) = 0 \]  

(\*)

FONCs for solution to (ii) with \( \alpha = \alpha_0 \) include:

\[ \frac{\partial L_{ii}}{\partial x} (\dot{x}(\alpha_0), \alpha_0; \dot{\lambda}(\alpha_0), \dot{\mu}(\alpha_0)) = \frac{\partial F}{\partial x} (\dot{x}(\alpha_0), \alpha_0) \]
\[ -\dot{\lambda}(\alpha_0) \frac{\partial g}{\partial x} (\dot{x}(\alpha_0), \alpha_0) - \dot{\mu}(\alpha_0) \frac{\partial h}{\partial x} (\dot{x}(\alpha_0), \alpha_0) = 0 \]

(**)

Rewriting (**) and using \( \dot{x}(\alpha_0) = x^*(\alpha_0) \):

\[ \frac{\partial F}{\partial x} (x^*(\alpha_0), \alpha_0) - \lambda^*(\alpha_0) \frac{\partial g}{\partial x} (x^*(\alpha_0), \alpha_0) \]
\[ -\left( \dot{\lambda}(\alpha_0) - \lambda^*(\alpha_0) \right) \frac{\partial g}{\partial x} (x^*(\alpha_0), \alpha_0) - \dot{\mu}(\alpha_0) \frac{\partial h}{\partial x} (x^*(\alpha_0), \alpha_0) = 0 \]

Using (\*) this implies:

\[ -\left( \dot{\lambda}(\alpha_0) - \lambda^*(\alpha_0) \right) \frac{\partial g}{\partial x} (x^*(\alpha_0), \alpha_0) - \dot{\mu}(\alpha_0) \frac{\partial h}{\partial x} (x^*(\alpha_0), \alpha_0) = 0 \]

\( \frac{\partial g}{\partial x} (x^*(\alpha_0), \alpha_0) \) and \( \frac{\partial h}{\partial x} (x^*(\alpha_0), \alpha_0) \) linearly independent \( \Rightarrow \dot{\lambda}(\alpha_0) = \lambda^*(\alpha_0), \dot{\mu}(\alpha_0) = 0 \).
By the envelope theorem:

\[
\frac{\partial F^*}{\partial \alpha}(\alpha_0) = \frac{\partial L_1}{\partial \alpha}(x^*(\alpha_0) ; \alpha_0 ; \lambda^*(\alpha_0)) - \frac{\partial F}{\partial \alpha}(x^*(\alpha_0) ; \alpha_0) - \frac{\partial g}{\partial \alpha}(x^*(\alpha_0) ; \alpha_0)
\]

and

\[
\hat{F}(\alpha_0) = \frac{\partial L_{ii}}{\partial \alpha}(\hat{x}(\alpha_0) ; \lambda(\alpha_0), \hat{\mu}(\alpha_0)) - \frac{\partial \hat{F}}{\partial \alpha}(\hat{x}(\alpha_0) ; \alpha_0)
\]

\[
= \frac{\partial F^*}{\partial \alpha} (\alpha_0)
\]

(using: \( \hat{x}(\alpha_0) = x^*(\alpha_0) \), \( \hat{\lambda}(\alpha_0) = \lambda^*(\alpha_0) \), \( \hat{\mu}(\alpha_0) = 0 \) ) Q.E.D.

b. Proof: Given \( z \in \mathbb{R}^n \), want to show:

\[
z' \left( \frac{\partial^2 \hat{F}}{\partial \alpha^2} (\alpha_0) - \frac{\partial^2 F^*}{\partial \alpha^2} (\alpha_0) \right) z \leq 0
\]

Consider \( \alpha = \alpha_0 + tz \) for \( t \in \mathbb{R}, t \neq 0 \). \( \alpha \neq \alpha_0 \) and, because \( A \) is open, \( \alpha \in A \) for \( t \) sufficiently small.

\[
\hat{F}(\alpha) \leq F^*(\alpha) \quad \text{(Imposing an additional constraint cannot increase the optimal value of } F(\cdot) \text{.)}
\]

Writing Taylor series expansions about \( \alpha_0 \):

\[
\hat{F}(\alpha_0) + \frac{\partial \hat{F}}{\partial \alpha}(\alpha_0)(\alpha - \alpha_0) + \frac{1}{2}(\alpha - \alpha_0)' \frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0 + \theta_1(\alpha - \alpha_0))(\alpha - \alpha_0)
\]

\[
\leq F^*(\alpha_0) + \frac{\partial F^*}{\partial \alpha}(\alpha_0)(\alpha - \alpha_0) + \frac{1}{2}(\alpha - \alpha_0)' \frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0 + \theta_2(\alpha - \alpha_0))(\alpha - \alpha_0)
\]

where \( \theta_1, \theta_2 \in (0,1) \). Using \( \hat{F}(\alpha_0) = F^*(\alpha_0) \) and \( \frac{\partial \hat{F}}{\partial \alpha}(\alpha_0) = \frac{\partial F^*}{\partial \alpha}(\alpha_0) \) from part a:

\[
\frac{1}{2} t^2 z' \frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0 + \theta_1(\alpha - \alpha_0)) z \leq \frac{1}{2} t^2 z' \frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0 + \theta_2(\alpha - \alpha_0)) z
\]
or \[
z^t \left( \frac{\partial^2 \hat{F}}{\partial \alpha^2} (\alpha_0 + \theta_1 (\alpha - \alpha_0)) - \frac{\partial^2 F^*}{\partial \alpha^2} (\alpha_0 + \theta_2 (\alpha - \alpha_0)) \right) z \leq 0
\]

Taking the limit at \( t \to 0 \) \((\alpha \to \alpha_0)\):

\[
z^t \left( \frac{\partial^2 \hat{F}}{\partial \alpha^2} (\alpha_0) - \frac{\partial^2 F^*}{\partial \alpha^2} (\alpha_0) \right) z \leq 0 \quad \text{Q.E.D.}
\]

2. The long- and short-run problems are:

\[
\max_{\text{w.r.t. } x_1, x_2} \ p f(x_1, x_2) - w_1 x_1 - w_2 x_2
\]

\[
\max_{\text{w.r.t. } x_1, x_2} \ p f(x_1, x_2) - w_1 x_1 - w_2 x_2 \quad \text{subject to } x_1 - c x_2 = 0
\]

where \( c \equiv \frac{x_1^* (\alpha_0)}{x_2^* (\alpha_0)} \) and \( \alpha_0 \equiv (p_0, w_{i0}, w_{20}) \).

Lagrangian: \( L(x, p, w_1, w_2; \lambda) = p f(x_1, x_2) - w_1 x_1 - w_2 x_2 - \lambda (x_1 - c x_2) \)

From problem 1: \( \frac{\partial^2 \hat{\pi}}{\partial \alpha^2} (\alpha_0) - \frac{\partial \pi^*}{\partial \alpha^2} (\alpha_0) \) is negative semi-definite. This implies non-positive diagonal elements, which implies, for \( i = 1, 2 \):

\[
\frac{\partial^2 \hat{\pi}}{\partial \alpha^2} (\alpha_0) - \frac{\partial \pi^*}{\partial \alpha^2} (\alpha_0) \leq 0 \quad (***)
\]

By the envelope theorem:

\[
\frac{\partial \pi^*}{\partial \alpha^2} (\alpha_0) = \frac{\partial \pi}{\partial \alpha^2} (x^* (\alpha_0), \alpha_0) = -x_i^* (\alpha_0)
\]

\[
\frac{\partial \hat{\pi}}{\partial \alpha^2} (\alpha_0) = \frac{\partial L}{\partial \alpha^2} (\hat{x}(\alpha_0), \alpha_0; \lambda(\alpha_0)) = -\hat{x}_i (\alpha_0)
\]

Substituting into (***): For \( i = 1, 2 \):

\[
-\frac{\partial \hat{x}_i}{\partial \alpha^2} (\alpha_0) \leq -\frac{\partial x_i^*}{\partial \alpha^2} (\alpha_0) \quad (****)
\]
Because \( \frac{\partial^2 \hat{\pi}}{\partial w^2_i} (\alpha_0), \frac{\partial^2 \pi^*}{\partial w^2_i} (\alpha_0) \geq 0 \) (For example, see Problem #2, Homework, F06),

\[ \frac{\partial \hat{x}_i}{\partial w_i} (\alpha_0), \frac{\partial x^*_i}{\partial w_i} (\alpha_0) \leq 0 \]

So (****) implies: \( \left| \frac{\partial \hat{x}_i}{\partial w_i} (\alpha_0) \right| \leq \left| \frac{\partial x^*_i}{\partial w_i} (\alpha_0) \right| \)

(SR factor demands no more elastic than LR factor demands.)

b. Now the SR problem is

\[ \max_{w.x_i. x_2} p f(x_1, x_2) - w_1 x_1 - w_2 x_2 \quad \text{subject to} \quad w_1 x_1 + w_2 x_2 = C \]

where \( C \equiv w_{10} x_1^*(\alpha_0) + w_{20} x_2^*(\alpha_0) \).

Lagrangian: \( L(x, p, w_1, w_2; \lambda) = p f(x_1, x_2) - w_1 x_1 - w_2 x_2 + \lambda (C - w_1 x_1 - w_2 x_2) \)

(***) is still valid, but now:

\[ \frac{\partial \hat{\pi}}{\partial w_i} (\alpha_0) = \frac{\partial L}{\partial w_i} (\hat{\lambda}(\alpha_0); \alpha_0) = -\left( 1 + \lambda(\alpha_0) \right) \hat{x}_i (\alpha_0). \]

Although \( \hat{\lambda}(\alpha_0) = 0 \), we typically won't have \( \frac{\partial \hat{\lambda}}{\partial w_i} (\alpha_0) = 0 \) so:

\[ \frac{\partial^2 \hat{\pi}}{\partial w^2_i} (\alpha_0) \neq -\frac{\partial \hat{x}_i}{\partial w_i}. \]

Restrictions on the curvature of the value function no longer relate directly to the slope of factor demands.