Econ 600
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Fall 2007

Homework solution outline

1. a. Lagrangians for the two problems:

$$L_i(x;\alpha;\lambda) = F(x;\alpha) - \lambda\, g(x;\alpha)$$

$$L_{ii}(x;\alpha;\lambda,\mu) = F(x;\alpha) - \lambda\, g(x;\alpha) - \mu\, h(x;\alpha)$$

FONCs for solution to (*i*) with $\alpha = \alpha_0$ include:

$$\frac{\partial L_i}{\partial x}\left(x^*(\alpha_0);\alpha_0;\lambda^*(\alpha_0)\right) = \frac{\partial F}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) - \lambda^*(\alpha_0)\frac{\partial g}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) = 0 \qquad (*)$$

FONCs for solution to (*ii*) with $\alpha = \alpha_0$ include:

$$\frac{\partial L_{ii}}{\partial x}\left(\hat{x}(\alpha_0);\alpha_0;\hat{\lambda}(\alpha_0),\hat{\mu}(\alpha_0)\right) = \frac{\partial F}{\partial x}\left(\hat{x}(\alpha_0);\alpha_0\right) - \hat{\lambda}(\alpha_0)\frac{\partial g}{\partial x}\left(\hat{x}(\alpha_0);\alpha_0\right) - \hat{\mu}(\alpha_0)\frac{\partial h}{\partial x}\left(\hat{x}(\alpha_0);\alpha_0\right) = 0 \qquad (**)$$

Rewriting (**) and using $\hat{x}(\alpha_0) = x^*(\alpha_0)$:

$$\frac{\partial F}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) - \lambda^*(\alpha_0)\frac{\partial g}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) - \left(\hat{\lambda}(\alpha_0) - \lambda^*(\alpha_0)\right)\frac{\partial g}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) - \hat{\mu}(\alpha_0)\frac{\partial h}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) = 0$$

Using (*) this implies:

$$-\left(\hat{\lambda}(\alpha_0) - \lambda^*(\alpha_0)\right)\frac{\partial g}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) - \hat{\mu}(\alpha_0)\frac{\partial h}{\partial x}\left(x^*(\alpha_0);\alpha_0\right) = 0$$

$\frac{\partial g}{\partial x}\left(x^*(\alpha_0);\alpha_0\right)$ and $\frac{\partial h}{\partial x}\left(x^*(\alpha_0);\alpha_0\right)$ linearly independent $\Rightarrow$ $\hat{\lambda}(\alpha_0) = \lambda^*(\alpha_0)$, $\hat{\mu}(\alpha_0) = 0$.

By the envelope theorem:

$$\frac{\partial F^*}{\partial \alpha}(\alpha_0) = \frac{\partial L_i}{\partial \alpha}\left(x^*(\alpha_0);\alpha_0;\lambda^*(\alpha_0)\right) = \frac{\partial F}{\partial \alpha}\left(x^*(\alpha_0);\alpha_0\right) - \lambda^*(\alpha_0)\frac{\partial g}{\partial \alpha}\left(x^*(\alpha_0);\alpha_0\right)$$

and

$$\frac{\partial \hat{F}}{\partial \alpha}(\alpha_0) = \frac{\partial L_{ii}}{\partial \alpha}\left(\hat{x}(\alpha_0);\alpha_0;\hat{\lambda}(\alpha_0),\hat{\mu}(\alpha_0)\right) = \frac{\partial F}{\partial \alpha}\left(\hat{x}(\alpha_0);\alpha_0\right)$$

$$-\hat{\lambda}(\alpha_0)\frac{\partial g}{\partial \alpha}\left(\hat{x}(\alpha_0);\alpha_0\right) - \hat{\mu}(\alpha_0)\frac{\partial h}{\partial \alpha}\left(\hat{x}(\alpha_0);\alpha_0\right)$$

$$= \frac{\partial F^*}{\partial \alpha}(\alpha_0) \quad \text{(using: } \hat{x}(\alpha_0) = x^*(\alpha_0),\ \hat{\lambda}(\alpha_0) = \lambda^*(\alpha_0),\ \hat{\mu}(\alpha_0) = 0\text{)} \quad \text{Q.E.D.}$$

b. <u>Proof:</u> Given $z \in \Re^m$, want to show:

$$z'\left(\frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0) - \frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0)\right)z \le 0$$

Consider $\alpha \equiv \alpha_0 + t\,z$ for $t \in \Re, t \ne 0$. $\alpha \ne \alpha_0$ and, because A is open, $\alpha \in A$ for t sufficiently small.

$\hat{F}(\alpha) \le F^*(\alpha)$ (Imposing an additional constraint cannot increase the optimal value of $F(\cdot)$.)

Writing Taylor series expansions about α_0:

$$\hat{F}(\alpha_0) + \frac{\partial \hat{F}}{\partial \alpha}(\alpha_0)(\alpha - \alpha_0) + \frac{1}{2}(\alpha - \alpha_0)'\frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0 + \theta_1(\alpha - \alpha_0))(\alpha - \alpha_0)$$

$$\le F^*(\alpha_0) + \frac{\partial F^*}{\partial \alpha}(\alpha_0)(\alpha - \alpha_0) + \frac{1}{2}(\alpha - \alpha_0)'\frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0 + \theta_2(\alpha - \alpha_0))(\alpha - \alpha_0)$$

where $\theta_1, \theta_2 \in (0,1)$. Using $\hat{F}(\alpha_0) = F^*(\alpha_0)$ and $\frac{\partial \hat{F}}{\partial \alpha}(\alpha_0) = \frac{\partial F^*}{\partial \alpha}(\alpha_0)$ from part *a*:

$$\frac{1}{2}t^2 z'\frac{\partial^2 \hat{F}}{\partial \alpha^2}(\alpha_0 + \theta_1(\alpha - \alpha_0))z \le \frac{1}{2}t^2 z'\frac{\partial^2 F^*}{\partial \alpha^2}(\alpha_0 + \theta_2(\alpha - \alpha_0))z$$

or $$z'\left(\frac{\partial^2 \hat{F}}{\partial \alpha^2}\left(\alpha_0+\theta_1\left(\alpha-\alpha_0\right)\right)-\frac{\partial^2 F^*}{\partial \alpha^2}\left(\alpha_0+\theta_2\left(\alpha-\alpha_0\right)\right)\right)z \le 0$$

Taking the limit at $t \to 0 \left(\alpha \to \alpha_0\right)$:

$$z'\left(\frac{\partial^2 \hat{F}}{\partial \alpha^2}\left(\alpha_0\right)-\frac{\partial^2 F^*}{\partial \alpha^2}\left(\alpha_0\right)\right)z \le 0 \qquad \text{Q.E.D.}$$

2. The long- and short-run problems are:

$$\max_{w.r.t.\ x_1,\ x_2} \ pf\left(x_1,x_2\right)-w_1x_1-w_2x_2$$

$$\max_{w.r.t.\ x_1,\ x_2} \ pf\left(x_1,x_2\right)-w_1x_1-w_2x_2 \quad \textit{subject to} \quad x_1-cx_2=0$$

where $c \equiv {x_1^*\left(\alpha_0\right)}\big/{x_2^*\left(\alpha_0\right)}$ and $\alpha_0 \equiv \left(p_0, w_{10}, w_{20}\right)$.

Lagrangian: $L\left(x; p, w_1, w_2; \lambda\right)= pf\left(x_1,x_2\right)-w_1x_1-w_2x_2-\lambda\left(x_1-cx_2\right)$

From problem 1: $\frac{\partial^2 \hat{\pi}}{\partial \alpha^2}\left(\alpha_0\right)-\frac{\partial \pi^*}{\partial \alpha^2}\left(\alpha_0\right)$ is negative semi-definite. This implies non-positive diagonal elements, which implies, for i = 1, 2:

$$\frac{\partial^2 \hat{\pi}}{\partial w_i^2}\left(\alpha_0\right)-\frac{\partial^2 \pi^*}{\partial w_i^2}\left(\alpha_0\right) \le 0 \qquad (***)$$

By the envelope theorem:

$$\frac{\partial \pi^*}{\partial w_i}\left(\alpha_0\right)=\frac{\partial \pi}{\partial w_i}\left(x^*\left(\alpha_0\right);\alpha_0\right)=-x_i^*\left(\alpha_0\right)$$

$$\frac{\partial \hat{\pi}}{\partial w_i}\left(\alpha_0\right)=\frac{\partial L}{\partial w_i}\left(\hat{x}\left(\alpha_0\right);\alpha_0;\lambda\left(\alpha_0\right)\right)=-\hat{x}_i\left(\alpha_0\right)$$

Substituting into (***): For i = 1, 2:

$$-\frac{\partial \hat{x}_i}{\partial w_i}\left(\alpha_0\right) \le -\frac{\partial x_i^*}{\partial w_i}\left(\alpha_0\right) \qquad (****)$$

Because $\frac{\partial^2 \hat{\pi}}{\partial w_i^2}(\alpha_0), \frac{\partial^2 \pi^*}{\partial w_i^2}(\alpha_0) \geq 0$ (For example, see Problem #2, Homework, F06),

$$\frac{\partial \hat{x}_i}{\partial w_i}(\alpha_0), \frac{\partial x_i^*}{\partial w_i}(\alpha_0) \leq 0 \quad \text{So (****) implies:} \quad \left|\frac{\partial \hat{x}_i}{\partial w_i}(\alpha_0)\right| \leq \left|\frac{\partial x_i^*}{\partial w_i}(\alpha_0)\right|$$

(SR factor demands no more elastic than LR factor demands.)

b. Now the SR problem is

$$\max_{w.r.t.\ x_1,\ x_2} pf(x_1, x_2) - w_1 x_1 - w_2 x_2 \quad \textit{subject to} \quad w_1 x_1 + w_2 x_2 = C$$

where $C \equiv w_{10} x_1^*(\alpha_0) + w_{20} x_2^*(\alpha_0)$.

Lagrangian: $L(x; p, w_1, w_2; \lambda) = pf(x_1, x_2) - w_1 x_1 - w_2 x_2 + \lambda(C - w_1 x_1 - w_2 x_2)$

(***) is still valid, but now:

$$\frac{\partial \hat{\pi}}{\partial w_i}(\alpha_0) = \frac{\partial L}{\partial w_i}\left(\hat{x}(\alpha_0); \alpha_0; \hat{\lambda}(\alpha_0)\right) = -\left(1 + \hat{\lambda}(\alpha_0)\right)\hat{x}_i(\alpha_0).$$

Although $\hat{\lambda}(\alpha_0) = 0$, we typically won't have $\partial \hat{\lambda} / \partial w_i (\alpha_0) = 0$ so:

$$\frac{\partial^2 \hat{\pi}}{\partial w_i^2}(\alpha_0) \neq -\frac{\partial \hat{x}_i}{\partial w_i}.$$

Restrictions on the curvature of the value function no longer relate directly to the slope of factor demands.